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Survey on Convex Optimization in Power Distribution Networks

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Abstract: *The purpose of this paper is to present a comprehensive survey of various optimization methods used to solve OPF problems.This paper is concerned with the minimum cost flow problem over an arbitrary flow network. Each node is, associated with some possibly unknown injection, each line has two unknown flows at its ends related to each other via a nonlinear function, and all injections and flows need to satisfy certain box constraints. This problem, named (GNF), due to its nonlinear equality constraints. Under the assumption of monotonicity and convexity of the flow and cost functions, a convex relaxation is proposed A primary application of this work is in optimization over power networks. Recent work on the (OPF) problem has shown that this non-convex problem can be solved efficiently using (SDP) after two approximations: relaxing angle constraints (by adding virtual phase shifters) and relaxing power balance equations to inequality constraints. The problem to minimize power losses in an electrical network subject to voltage and power constraints is in general hard to solve.However, it has recently been discovered that SDP relaxations in many cases enable exact computation of the global optimum. A fundamental reason for the successful relaxations, namely that the passive network components give rise to matrices with non-negative off diagonal entries. Recent progress on quadratic programming with Metzler matrix structure can therefore be applied.The objective of an Optimal Power Flow (OPF) algorithm is to find steady state operation point which minimizes generation cost, loss etc. or maximizes social welfare, loadability etc. while maintaining an acceptable system performance in terms of limits on generators' real and reactive powers, line flow limits, output of various compensating devices.*

Keywords: *Optimal power flow, lossynetwork, power distribution network. convex optimization, Semidefinite programming (SDP) method, convex relaxation, global optimization.*

I. Introduction

Power flow has been studied since the 1930's. Earlier studies were done using dc network analyzer [1], in which the current flow was proportional to the power flow. Mathematically, this represent a linearization of the power flow equations in[(2] and is called the DC power flow. The optimal power flow problem was proposed by [3], and have received a tremendous amount of attention since then (for example, see [4,5] for surveys) Despite the amount of research effort, OPF remains a challenge. In practice, it is generally solved using Newton-type method, which convergences to a local optimum (although convergence is not guaranteed). Some non-iterative methods have been proposed [6, 7], which may have wider radius of convergences compared to the Newton-type methods.

II. Convex Optimisation Technques

Mathematical programming approaches to power systems problems came to prominence in the 1960's and 1970's, primarily in the context of designing transmission networks [8] and power. The optimal power flow problem was first discussed in Carpentier's paper [9] in 1962. The objective of an Optimal Power Flow (OPF) algorithm is to find an optimal operating point, which minimizes the generation cost or network loss, subject to a wide range of practical constraints, e.g. bus voltage limits, bus power limits, thermal line constraints, etc. The OPF problem is a non-convex and challenging for the following two reasons [10]. Firstly, since the injected power at buses depends quadratically one the voltages at the buses, the optimization problem is non-linear. Secondly, power system need to satisfy a series of constraints such as active/reactive power balance equations, power flow limit of line, bus voltage magnitude limits and active/reactive power generation limits. Given the practical importance of the problem, a great many studies have been developed to give efficient solution methods, including linear programming, non-linear programming, quadratic programming, interior point methods, Lagrangian relaxation, artificial intelligence, fuzzy logic, evolutionary programming, genetic algorithm and particle swarm optimization [11], [12]. One widely used for method is the DC (direct current)-OPF, which linearized the OPF problem with assumptions that the power line is lossless, the voltage magnitudes are fixed and the voltage angles are small [13]. This method is not accurate and will perform poorly if the resistance/inductance ratio of the line is high. In an effort to convexity the AC OPF problem, various convex relaxation techniques have been developed. Semidefinite programming (SDP) method can create a convex relaxation of the OPF problem. In [15], it proposes solving the Lagrangian dual problem instead of solving the OPF problem directly. Relaxation methods are another class of algorithm for OPF. These algorithm converts the non-convex OPF to a convex problem through different types of relaxations. For general networks, Lagrangian dual relaxations was proposed in [16,17] conic relaxation was proposed in [18], and semidefinite programming (SDP) relaxation was proposed in [19]. The SDP relaxation technique was explored in-depth in [20] to show that the relaxation is tight for the IEEE benchmarks in [21]. Since the relaxations are convex, they can be solved by Newton or interior point methods in polynomial time [22]. The SDP relaxation is exact if and only if the duality gap is zero. More importantly, [23] makes the observation that OPF has a zero duality gap for IEEE benchmark systems with 14, 30, 57, 118 and 300 buses, in addition to several randomly generated power networks. This technique is the first method proposed since the introduction of the OPF problem, which is able to find a provably global solution for practical OPF problems. The SDP relaxation for OPF has attracted much attention due to its ability to find a global solution in polynomial time, and it has been applied to various applications in power systems including: voltage regulation in distribution systems [24], state estimation [25], and calculation of voltage stability margin [26], economic dispatch in unbalanced distribution networks [27], and power

Management under time-varying conditions [28]. The SDP relaxation is exact in two cases: (i) for acyclic networks, (ii) for cyclic networks after relaxing the angle constraints. This exactness was related to the passivity of transmission lines and transformers. A question arises as to whether the SDP relaxation remains exact for mesh (cyclic) networks (without any angle relaxations). The relaxation is not exact even for a three-bus cyclic network. Motivated by this negative result, aim is to explore the limitations of the SDP relaxation for mesh networks. More precisely, there are four (almost) equivalent ways to model the capacity of a power line but only one of these models gives rise to the exactness of the SDP relaxation. Furthermore, substantiatethat with this type of network which has a convex injection region in the lossless case and a non-convex injection region with a convex Pareto front in the lossy case. The importance of this result is that the SDP relaxation works on certain cyclic networks, for example the ones generated from three-bus subgraphs (this type of network is related to three-phase systems). In the case when the SDP relaxation does not work, an upper bound is provided on the rank of the minimum-rank solution of the SDP relaxation. This bound is related only to the structure of the power network and this number is expected to be very small for real-world power networks. Finally, a heuristic method is proposed to enforce the SDP relaxation to produce a rank-1 solution for general networks (by somehow killing the undesirable eigenvalues of the low-rank solution). Boundary of a convex set.

The “network flow” problem is of significant importance in computer science, operation research, and engineering [28,29]. This problem has immediate applications in communication networks, power and commodity distribution, financial budgeting, and production scheduling and assignment, among other fields. The minimum-cost flow problem aims to find optimal flows in a given network such that the overall cost of production and/or transportation is minimized. In this problem, the network is used to carry some commodity of interest between pre-specified sources and destinations. To formalize a flow network, consider a graph consisting of nodes and lines. There is an injection of some commodity at every node, and there are two flows over each line. One flow enters the line from an endpoint and the second flow leaves from the other endpoint. Depending on the sign of its injection, each node can be considered as a supplier or consumer. This problem was developed and solved in [30] for lossless networks. Although the algorithm proposed in [31] is efficient, it does not apply to certain real-world networks because the line losses are ignored. More precisely, the flow entering a line may not be equal to the outgoing flow in practice. Driven by this practical consideration, the lossy network flow problem has drawn much attention. a generalized network (also known as network with gain) in which each outgoing flow is proportionally related to the entering flow via a constant gain. This type of network flow problem has been studied extensively [31], [32]. Assuming that the cost functions are convex, this type of lossy network can be solved in polynomial time (up to a given accuracy) because of the convex nature of its objective and constraints [33]. Recently, [34] has introduced a more general network flow problem, referred to as Generalized Network Flow (GNF). In GNF, the output flow over each line is a nonlinear function of the input flow. This is motivated by the fact that the line losses are nonlinear in certain real-world networks, such as electrical power networks [35]. Assume that the cost and flow functions are all monotonic and convex, which is a fairly reasonable assumption in practice. The GNF problem is highly non-convex due to its nonlinear equality constraints. However, relaxing the equality constraints into convex in- equality leads to a convex relaxation of the problem, named convexified generalized network flow (CGNF). The work [36] has proved that this relaxation is exact for the optimal injections but may not yield feasible (optimal) flows for GNF. Since the optimal injections for GNF can systematically be found using CGNF, the main objective of this paper is to study the possibility of finding optimal flows. First, we prove that if the optimal injection vector is a Pareto point in its feasible region, CGNF finds optimal flows for GNF. Second, we substantiate that the flow network can be divided into two sub-networks such that: (i) CGNF obtains optimal flows over one sub-network, (ii) the lines between the two sub-networks are all congested at optimality and CGNF correctly identifies these lines. In other words, we relate the possible failure of CGNF in finding optimal flows for the whole network to certain congested lines. Moreover, we fully characterize the set of all optimal flow vectors. this set may be infinite, non-convex, and disconnected, This is concerned with a fundamental re- source allocation problem for electrical power networks. A convex relaxation based on semidefinite programming (SDP) is able to find a global solution of OPF for IEEE benchmark systems, and moreover this technique is guaranteed to work over acyclic (distribution) networks. the potential of the SDP relaxation for OPF over cyclic (transmission) networks. Given an arbitrary weakly-cyclic network with cycles of size 3, it is shown that the injection region is convex in the lossless case and that the Pareto front of the injection region is convex in the lossy case. It is also proved that the SDP relaxation of OPF is exact for this type of network. Moreover, it is shown that if the SDP relaxation is not exact for a general mesh network, it still has a low-rank solution whose rank depends on the structure of the network. Finally, a heuristic method is proposed to recover a rank-1 solution for the SDP relaxation whenever the relaxation is not exact.

A global optimization technique for a broad class of nonlinear optimization problems, including quadratic and polynomial optimization problems. The main objective of this paper is to investigate how the (hidden) structure of a given real/complex- valued optimization problem makes it easy to solve. To this end, three conic relaxations are proposed. Necessary and sufficient conditions are derived for the exactness of each of these relaxations, and it is shown that these conditions are satisfied if the optimization problem is highly structured. More precisely, the structure of the optimization problem is mapped into a generalized weighted graph, where each edge is associated with a weight set extracted from the coefficients of the optimization problem. In the real-valued case, it is shown that the relaxations are all exact if each weight set is sign definite and in addition a condition is satisfied for each cycle of the graph. It is also proved that if some of these conditions are violated, the relaxations still provide a low-rank solution for weakly cyclic graphs. In the complex-valued case, the notion of “sign definite complex sets” is introduced for complex weight sets. It is then shown that the relaxations are exact if each weight set is sign definite (with respect to complex numbers) and the graph is acyclic. Three other structural properties are derived for the generalized weighted graph in the complex case, each of which guarantees the exactness of some of the proposed relaxations. This result is also generalized to graphs that can be decomposed as a union of edge-disjoint subgraphs, where each subgraph has certain structural properties. As an application, it is proved that a relatively large class of real and complex optimization problems over power networks are polynomial-time solvable (with an arbitrary accuracy) due to the passivity of transmission lines and transformers.

Several classes of optimization problems, including polynomial optimization problems and quadratically constrained quadratic programs (QC- QPs) as a special case, are nonlinear/non-convex and NP-hard in the worst case. The paper [37] provides a survey on the computational complexity of optimizing various classes of continuous functions over some simple constraint sets. Due to the complexity of such problems, several convex relaxations based on semidefinite programming (SDP) and second-order cone programming (SOCP) have gained popularity [37,38]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact value or a lower bound on the optimal objective value. The paper [39] shows how SDP relaxation can be used to find better approximations for maximum cut (MAX CUT) and maximum 2-satisfiability (MAX 2SAT) problems. Another approach is proposed in [40] to solve the max-3-cut problem via complex SDP. The SDP relaxation converts an optimization problem with a vector variable to a convex optimization problem with a matrix variable, via a lifting technique. The exactness of the relaxation can then be interpreted as the existence of a low- rank (e.g., rank-1) solution for the SDP relaxation. Several papers have studied the existence of a low-rank solution to matrix optimization problems with linear matrix inequality (LMI) constraints [41, 42]. The papers [43] and [44] provide an upper bound on the lowest rank among all solutions of a feasible LMI problem. A rank-1 matrix decomposition technique is developed in [45] to find a rank-1 solution whenever the number of constraints is small. This technique is extended in [46] to the complex SDP problem. This is motivated by the fact that real-world optimization problems are highly structured in many ways and their structures could in principle help reduce the computational complexity. For example, transmission lines and transformers used in power networks are passive devices, and as a result optimization problems defined over electrical power networks have certain structures which distinguish them from abstract optimization problems with random coefficients. The high-level objective of this paper is to understand how the computational complexity of a given nonlinear optimization problem is related to its (hidden) structure. This is concerned with a broad class of nonlinear real/complex optimization problems, including QCQPs. The main feature of this class is that the argument of each objective and constraint function is quadratic (as opposed to linear) in the optimization variable and the goal is to use three conic relaxations (SDP, reduced SDP and SOCP) to convexify the argument of the optimization problem. In this work, the structure of the nonlinear optimization problem is mapped into a generalized weighted graph, where each edge is associated with a weight set constructed from the known parameters of the optimization problem (e.g., the co- efficients). This generalized weighted graph captures both the sparsity of the optimization problem and possible patterns in the coefficients. First, it is shown that the proposed relaxations are exact for real-valued optimization problems, provided a set of conditions is satisfied. These conditions need each weight set to be sign definite and each cycle of the graph to have an even number of positive weight sets. It is also shown that if some of these conditions are not satisfied, the SDP relaxation is guaranteed to have a rank-2 solution for weakly cyclic graphs, from which an approximate rank-1 solution may be recovered. To study the complex-valued case, the notion of "sign-definite complex weight sets" is introduced and it is then proved that the relaxations are exact for a complex optimization problem if the graph is acyclic with sign definite weight sets (with respect to complex numbers). The complex case is further studied and it is proved that the SDP relaxation is tight for four types of graphs as well as any acyclic combination of these types of graphs. As an application, it is also shown that a large class of energy optimization problems may be convexified due to the physics of power networks..

Consider an arbitrary power network with PV and PQ buses, where active powers and voltage magnitudes are known at PV buses, and active and reactive powers are known at PQ buses. The classical power flow (PF) problem aims to find the unknown complex voltages at all buses. This problem is usually solved approximately through linearization or in an asymptotic sense using Newton's method, given that the solution belongs to a good regime containing voltage vectors with small angles. The question arises as to whether the PF problem can be cast as the solution of a convex optimization problem over that regime. More precisely, a class of convex optimization problems with the property that they all solve the PF problem as long as angles are small. Each convex problem proposed in this work is in the form of a semidefinite program (SDP). Associated with each SDP, we explicitly characterize the set of complex voltages that can be recovered via that convex problem. Since there are infinitely many SDP problems, each capable of recovering a potentially different set of voltages, designing a good SDP problem is cast as a convex problem.

In [47], it shows that the load flow problem of a radial distribution system (tree networks) is a convex problem and can be modeled in the form of a conic program. However, the result could not be applied to a meshed network. Then the question lies on what kind of networks the OPF problem can be convexified. Power system consists of transmission networks and distribution networks. The transmission network is usually made up of high to very high voltage lines that designed to transfer power from major generators to areas in need, the networks' voltages are typically above 100 kV. Distribution networks is designed to distribute power from the transmission network to end users, it is usually made up of low voltage lines with voltage magnitudes below 100 kV. Traditionally OPF problem mainly focus on transmission networks, but nowadays with increasing interest on renewable energy, distributed generation and smart grid, comes with increasing demand on solving the OPF problem in distribution networks. Focus will be on topics about convex optimization in distribution networks. There are typically two types of distribution networks, radial (tree network) or interconnected network. A tree network leaves the station and passes with no normal connection to any other supply. This is typical of long rural lines. An interconnected network is generally found in urban areas and has multiple connections to other points of supply. Since most distribution networks is with a tree topology and research on tree networks will shed light on the general problem, the goal of this is to study on the tree topology of distribution networks.

III. Conclusions

This paper summarizes Convex Optimization Techniques for optimal power flow in complicated distribution networks. Recent development suggests great potential in these approaches of Power system control. Aside from specific questions, more general concern of distributed strategies relate private issues, robustness to communication uncertainties and failures.

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