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A Review on the action of Hecke Operators on Drinfeld Modular forms

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Abstract: *In this review research paper we evaluate the explicit construct of the semisimple section of the Hecke algebra that operates on Drinfeld modular form of complete level modulo T. We demonstrate that modulo T the Hecke algebra has a non-zero semisimple section. On the other hand a popular theorem of Serre asserts that the traditional modular varieties the action of T_l for any peculiar prime l is nilpotent modulo 2. After demonstrating outcome for Drinfeld modular forms modulo T. We use computations of the Hecke operation modulo T, demonstrating that certain forces of the Drinfeld modular form h can't be Eigen forms. Finally we conjecture that the Hecke algebra that functions on Drinfeld modular forms of full level is no soft for large weight which again the impact of traditional situation.*

Keywords: *Drinfeld modular forms,Hecke operator, reduction of Drinfeld modular forms modulo T.*

I. Introduction

NilpotenceoftheHeckealgebramodulo2forthefullmodulargroupSL$_2$((ℤ)(integer))wasinitiallylookbySerre [1,§5].Theoutcomehasmanyvaluablearithmeticuses(seeSec.2.7andCh.5of[2]).Just,NicolasandSerre[3, 4]havefoundtheorderofnilpotenceandthe makingoftheHeckealgebramodulo2forfullmodulargroup.OnoandTaguchi[5]havelearnedmorespacesofmodularformsforwhichtheHeckeactionisnilpotentmodulo2andhavegivenout comesofthenilpotencefor quadraticforms,divide functionsandcentralvaluesoftwistedmodularL-functions.Thebasicaimofthe currentreview workistolearnthemakingoftheHeckealgebrathatoperatesonDrinfeldmodularformsforGL$_2$(𝔽q [T])andtheirreductionsmoduloT.

Last, wegiveexample thatmakesitnaturaltoshowConclusion,whichrelatedtheprobablenon-smoothnessoftheHeckealgebraovertherationalfunctionfield.Wetargetthattheresulttoourconclusionwillbeaffirmative,whichwillsupposethattheHeckealgebraovertherationalfunctionfieldisnotsmooth.Classical theHeckealgebrathatoperatesonthespaceofmodularformsoffulllevelisover ℚ,thereforeapositiveconclusiontoourquestionwillunderlineotherdifferencebetweentheconstructionoftheHeckealgebrainthecaseofclassical modular formsandinthecaseofDrinfeldmodularforms.

II. Hecke algebra modulo T

In this part we use the structure of $\hat{H}^{ss}_{k,m}$ = $\mathbb{F}q$ $[(\mathbb{F}q^*)^m]$completely. We will use of theSerre Derivatives, which we are as follows.
Define,

$$\Theta := \frac{1}{\tilde{\pi}}\frac{d}{dz} = -t^2\frac{d}{dt}.$$

The operator Θ does not preserve the order of $\mathcal{M}$,but the $\mathcal{K}^{th}$ Serre derivative are

$$\partial k := \Theta + k\mathrm{E}$$

that can show (see [6, (8.5)]) such that$\partial k: M_{k,m} \rightarrow M_{k+2,m+1}$
andfor that

$$k = k_1 + k_2, \qquad f_1, f_2$$

Drinfeldmodularformsofweightsk_1, k_2,respectively,
thenweget

$$\partial_k(f_1 f_2) = \partial_{k1}(f_1)f_2 + f_1\partial_{k2}(f_2)$$

wehave

$$\partial_{q+1}\, h = 0$$

which,togetherwiththe help of $E \equiv h \bmod T$,which showsthatOperatorpreserves$\widetilde{\mathcal{M}}$.
Indeed

$$\partial_{q+1}\, h=0$$

$\Rightarrow \Theta h \equiv -h^2 \bmod T$.

Aswe know that, let𝔭standforeitheraprimeidealof$A$differentfrom(T)orforthemonicgeneratorofsuchanideal.Thereductionof𝔭modulo T,that is,theconstanttermof𝔭,willbedenotedby$\tilde{\mathfrak{p}}$.
Byusing the equation

$$\frac{d}{dz}\mathcal{T}_{\mathrm{p,k}} f = \mathfrak{p}^{k} f(\mathfrak{p}z) + \sum_{\beta\in S\mathfrak{p}} f\left(\frac{z+\beta}{\mathfrak{p}}\right)$$

we get

$$\frac{d}{dz}\mathcal{T}_{\mathrm{p,k}} f = \mathfrak{p}^{k+1} f'(\mathfrak{p}z) + \frac{1}{\mathfrak{p}}\sum_{\beta\in S\mathfrak{p}} f'\left(\frac{z+\beta}{\mathfrak{p}}\right)$$

This in terms of operators shows that

$$\mathfrak{p}\cdot\Theta\mathcal{T}_{\mathfrak{p},k} = \mathcal{T}_{\mathfrak{p},k+2}\Theta$$

III. Theorem

Consider𝔭be a prime different from. Consider$n$be a non-negative (-ve) integer.Then

$$\mathcal{T}_{\mathrm{p}} h^n \equiv \tilde{\mathfrak{p}}^n h^n + (\textit{lower ordertermsin h}) \bmod T.$$

Proof:-
Claim: -We want toprovethat

$$\mathcal{T}_{\mathrm{p}} h^n \equiv \tilde{\mathfrak{p}}^n h^n + (\textit{lower ordertermsin h}) \bmod T.$$

By backward substitutionon$n$.
Consider n begivenandletp^{ν} bethesmallestpowerof$p$biggerthanorequalto n.
If$0 \le n \le p$,thenwealreadymentioned in the equation

$$\mathcal{T}_{\mathrm{p,q-1}} g = \mathfrak{p}^{q-1} g \quad \text{and}$$

$$\mathcal{T}_{\mathrm{p,j(q+1)}} h^j \equiv \mathfrak{p}^j h^j$$

sothat

$$\mathcal{T}_{\mathrm{p}} g = p^{q-1} g$$

$$\mathcal{T}_{\mathrm{p}} h^n \equiv \mathrm{p}^n h^n (1 \le n \le p),$$

andtherefore

$$\mathcal{T}_{\mathrm{p}} \tilde{g} = \mathcal{T}_{\mathrm{p}} 1 = \tilde{\mathfrak{p}}^{q-1} = 1$$

$$\mathcal{T}_{\mathrm{p}} \tilde{h}^n \equiv \tilde{\mathfrak{p}}^n \tilde{h}^n (1 \le n \le p),$$

Thisprovestheresultfor $1 \le n \le p$.
Supposethat$\nu>1$,that is,$p^{\nu-1} < n \le p^{\nu}$.
Assumethattheresultistruefor$n_0$intherange$1 \le n_0 \le p^{\nu-1}$.
If$p|n$,then$n = pn_0$for$n_0$between$p^{\nu-2}$and$p^{\nu-1}$andbytheinductionhypothesis

$$\mathcal{T}_{\mathrm{p}} h^n = (\mathcal{T}_{\mathrm{p}} h^{n_0})^{\mathrm{p}} \equiv (\tilde{p}^{n_0} h^{n_0})^{\mathrm{p}} + (\text{lowerordertermsin h mod } T)$$

If$p \nmid n$,thenwrite

$$\mathcal{T}_{\mathrm{p}} h^n = \varepsilon_n h^n + (\textit{lower order terms in } h \bmod T)$$

Weapply$\tilde{p}\Theta$totheequationand use the equation as follows

$$\tilde{p}\cdot \Theta\mathcal{T}_{\mathrm{p}},k \equiv \mathcal{T}_{\mathrm{p,k+2+s(q-1)}}\Theta \bmod T,$$

$$\mathcal{T}_{\mathrm{p}}(\Theta\, h^n)\equiv\tilde{p}.\Theta(\mathcal{T}_{\mathrm{p}}h^n) \equiv \tilde{p}.\Theta(\varepsilon_n h^n + (\textit{lower order terms in } h \bmod T))$$

By usingthe following equation

$$\Theta h \equiv -h^2 \bmod T.$$

then

$$\mathcal{T}_{\mathrm{p}}(-nh^{n+1})\equiv -n\tilde{p}\varepsilon_n.h^{n+1} + (\textit{lower order terms in } h \bmod T)$$

As$p \nmid n$wehave

$$\mathcal{T}_{\mathrm{p}}(h^{n+1}) \equiv \tilde{p}\varepsilon_n.h^{n+1} + (\textit{lower order terms in } h \bmod T)$$

Theequationaboveshowsthatonecanprovetheresultfornifoneassumestheresultforn+1and$p \nmid n$ .Thisfinishestheproofastheresultfor$p^{\nu-1}$isdeducedfromtheoneforp^{ν}(whichwealreadyhavededucedfromtheresult fortherange$p^{\nu-2}$to $p^{\nu-1}$),theresultfor$p^{\nu-2}$isdeducedfromtheonefor$p^{\nu-1}$andsoon.

Example:-Let$q$=2.Sincethetypeisdeterminedmodulo$(q-1)$,weknowthat$m$=0.Thefollowingtableshowsthat thefirstweights$k$forwhichtheminimalpolynomialof$\mathcal{T}$pisnotseparableforeveryprimepofdegree≤5:

k	9	13	15	16	17	18	19	21	23	24	25	26	27	28
$dim_K M_{k,0}$	4	5	6	6	6	7	7	8	8	9	9	9	10	10

Wenotethatfor$k \geq 3$,$M_{k,0}$hastwoone-dimensionalHeckeinvariantsubspaces(thespaceofEisensteinseriesandthespaceofsingle-cuspidalforms),thereforethepace$M_{9,0}$isactuallythefirstspaceforwhichtheHeckeactioncanbeinseparable.

IV. Conclusion

Given$q$,dothereexist$k$>>0and$m$,0≤ m<q−1,k≡2$m$mod(q−1),suchthat $H^2(\mathcal{H}_{k,m}, M_{k,m}) \neq 0$
Inparticular,iftheanswertothequestionis ‘Yes’,thenthisimpliesthat$H_{k,m}$isnotsmoothforsome$k$>>0andsome$m$.

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