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Simulation and Linear Stability Analysis of Mathematical Model for Bacterial Growth

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Abstract: *Linear Stability analysis plays a major role in understanding the efficiency of mathematical models. The analytical methods and numerical methods can be used effectively to compare the accuracy of the model. In this paper we will investigate the model using linear stability analysis. In particular we find equilibrium of the model, analyze the behaviour of the system in the vicinity of the equilibrium, and investigate the behaviour of the equilibrium when parameter values change especially when it concerns stability properties. We will also investigate the model numerically, that includes writing a MATLAB function that computes the right-hand side of the dynamical equations given by Bacterial Growth Model in question. Using rk6 and Multistep methods we explore the numerical solutions of the model and how they depend on the initial conditions and parameter values. For some fixed value of parameter(s). Then we compare the solutions obtained from the linear stability analysis method to the numerical solution starting from the same initial condition in the vicinity of an equilibrium point. We have also written MATLAB codes for the two new methods.*

Keywords *-Linear Stability Analysis, Runge-Kutta 6th order method, numerical methods, Eigen values, Eigen vectors, Dynamic Modelling, Simulation of mathematical model*

I. Introduction

The Modelling of bacterial growth into a mathematical model plays a vital role in Bio-Medical field and checking the stability of such models is essential and necessary. It decides the accuracy of the constructed model. In Section II of this paper we explain what Linear Stability Analysis is. In section III we will investigate the model using linear stability analysis. In particular to find equilibrium of the model, analyse the behaviour of the system in the vicinity of the equilibrium, investigate the behaviour of the equilibrium when parameter(s) values Change. In section IV we will investigate the model numerically, that includes simulating the model using a MATLAB function that computes the right-hand side of the dynamical equations given by the model in question. Then we derive and use rk6 to explore the numerical solutions of the model and how they depend on the initial conditions and parameter values. For some fixed value of parameter(s), comparing the solutions obtained from the analytical method to the numerical solution starting from the same initial condition in the vicinity of an equilibrium point. We use MATLAB codes for the two methods, one-step and multistep numerical solutions for our model. We also construct the step size-error diagrams for rk6 and Multistep Method to determine the order of the methods [1]. We use the solution with a small step size as the “exact” reference solution.

Many research papers were published for the development of mathematical models for Bacterial growth and they were successful in establishing the relationship between the growth rate and population count. The linear stability analysis of these models will give an upper hand in proving the accuracy of these models. Following are few of the mathematical models for bacterial growth. Kapur, J.N. and Khan, Q.J.A. [3] gave two models based on consideration of enzyme kinetics and compares them with existing model and obtained S shaped curve with limiting population growth and general point of inflexion. Baranyi, J., Roberts, T.A., and McClure, P. [4] described the bacterial growth by a non -autonomous differential equation. In addition they showed a possible way to apply this theory in food microbiology. Juskat, A., Gedminiene, G., Ivanec, R. [5] presented a model which is expressed symbolically as a finite combination of elementary functions. This approach can be applied in other areas of modern biology such as dynamics of various cellular processes and enzyme and receptor kinetics. Teleken, J.T., Robazza, W., Gomes, G., [6] proposed and evaluated mathematical model that predicts a microbial growth in a dairy products. In addition, their model provides equations for the evaluation of the maximum specific growth rate and the duration of the lag phase. Liu, Q., Jiang, D., Shi, N., [7] presented a paper which concerns with the dynamical behaviour of a stochastic SIQR epidemic model with standard incidence which is disturbed by both white and telegraph noises. In addition to that some numerical simulations are introduced to demonstrate the analytical results. Liu, C., Yu, L., Zhang, Q., and Li, Y.[8] investigated existence and uniqueness of global positive solution for stochastic system with double time delays, they also studied asymptotic behaviour of the interior equilibrium by constructing appropriate Lyapunov functions. Vidurupola, S. [9] investigated an extended deterministic lytic bacteriophage model that contained phage –resistant bacteria, bacteria complexes and bacteria debris was formulated, analyzed and numerically simulated. She also suggested that inclusion of these new states provides more realism and an accurate assessment of the bacteriophage interaction.

II. What Is Linear Stability Analysis

In Mathematics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions [2,9,10,11]. The heat equation, for example, is a stable partial differential equation because small perturbations of initial data lead to small variations in temperature at a later time as a result of the maximum principle. In the theory of differential equations and dynamical systems, a particular stationary or quasi-stationary solution to a nonlinear system is called linearly unstable if the linearization of the equation at this solution has the form $\frac{dr}{dt} = Ar$, where A is a linear operator whose spectrum contains eigenvalues with positive real part.

a. Linearization [2, 12]:

Linearization is finding the approximation to a function at a given point. In the study of dynamical system linearization is a method for assessing the local stability of an equilibrium point of a system of nonlinear Diff. Equations or discrete dynamical system. Linearization makes it possible to use tools for studying linear systems to analyse the behaviour of a nonlinear function near a given point.

b. Stability Analysis:

In stability analysis of autonomous system, one can use Eigen values of the Jacobian matrix evaluated at a hyperbolic equilibrium point to determine the nature of that equilibrium.

c. Procedure to determine stability of a system:

1. Determine all the stationary points of the given system in order to find equilibria.
2. Compute all Partial Differentials Equations of the right hand side of the given system and construct the Jacobian matrix.
3. Evaluate the Jacobian matrix at the steady state.
4. Compute Eigen values.
5. Conclude stability or instability based on the real parts of the Eigen values.

d. Stability Classification:

1. If the Eigen values of the Jacobian matrix all have real parts less than *Zero* then the state is *Stable*.
2. If at least one of the Eigen value of the Jacobian matrix has real part greater than *Zero* then the state is *Unstable*
3. If Eigen values have different signs then the state is a Saddle point; saddle point is always *Unstable*.

III. Analysis of the Linear Stability of the given Bacterial Model

The System is given by, [2]

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x^2y}{(\propto^2+x^2)} + x(1-x) \\ \frac{dy}{dt} &= -\varepsilon y + \frac{x^2y}{(\propto^2+x^2)} \end{aligned} \tag{A}$$

To Analyse the given system we must first find out the stationary points to find its equlibria.For that consider the following system of equations.

$$-\frac{x^2y}{(\propto^2+x^2)} + x(1-x) = 0 \tag{1}$$

$$-\varepsilon y + \frac{x^2y}{(\propto^2+x^2)} = 0 \tag{2}$$

From (1) We have,

$$x\left(-\frac{xy}{(\propto^2+x^2)} + (1-x)\right) = 0$$

$$\Rightarrow x = 0 \quad or \quad \left(-\frac{xy}{(\propto^2+x^2)} + (1-x)\right) = 0$$

$$\Rightarrow x = 0 \tag{a}$$

$$\left(-\frac{xy}{(\propto^2 + x^2)} + (1-x)\right) = 0 \qquad \text{(b)}$$

From (2) we have,

$$y\left(-\varepsilon + \frac{x^2}{(\propto^2 + x^2)}\right) = 0$$

$$\Rightarrow y = 0 \ or \ \left(-\varepsilon + \frac{x^2}{(\propto^2 + x^2)}\right) = 0$$

$$\Rightarrow y = 0 \qquad \text{(c)}$$

$$\left(-\varepsilon + \frac{x^2}{(\propto^2 + x^2)}\right) = 0 \qquad \text{(d)}$$

If the sytem given by (1) and (2) is to satisfy R.H.S then we must have $(a \vee b) \wedge (c \vee d) = 0$. i.e.$(a \wedge c) \vee (a \wedge d) \vee (b \wedge c) \vee (b \wedge d) = 0$.

1) $(a \wedge c) = 0$ gives, $x = 0, \ y = 0$.

Therefore the first stationary point is (0,0).

2) $(a \wedge d) = 0$ gives,$x = 0, \ \frac{x^2}{(\alpha^2 + x^2)} = \varepsilon$.

3) $(b \wedge c) = 0$ gives,

$\left(-\frac{xy}{(\alpha^2 + x^2)} + (1-x)\right) = 0$ and$y = 0$

Substitute$y = 0$ in the above equation to get $x = 1$

Therefore the second stationary point is (1,0).

4) $(b \wedge d) = 0$ gives,

$\left(-\frac{xy}{(\alpha^2 + x^2)} + (1-x)\right) = 0$ and $\left(-\varepsilon + \frac{x^2}{(\alpha^2 + x^2)}\right) = 0$

i.e. $-\varepsilon(\propto^2 + x^2) + x^2 = 0, -\varepsilon \propto^2 + (1-\varepsilon)x^2 = 0$,

$(1-\varepsilon)x^2 = \varepsilon\alpha^2, x^2 = \frac{\varepsilon\alpha^2}{(1-\varepsilon)}, x = \sqrt{\frac{\varepsilon\alpha^2}{(1-\varepsilon)}}, \ x = \alpha\sqrt{\frac{\varepsilon}{(1-\varepsilon)}}$

Substitute in the other equation we get,

$$\frac{\varepsilon\, y}{x} = (x-1),$$

$$y = \frac{x}{\varepsilon}(x-1),$$

$$y = \frac{\alpha\sqrt{\frac{\varepsilon}{(1-\varepsilon)}}}{\varepsilon}\left(\alpha\sqrt{\frac{\varepsilon}{(1-\varepsilon)}} - 1\right),$$

on simplification we get

$$y = \frac{\alpha^2}{1-\varepsilon} - \frac{\alpha}{\sqrt{\varepsilon\sqrt{1-\varepsilon}}}, \ x = \alpha\sqrt{\frac{\varepsilon}{(1-\varepsilon)}}$$

Therefore the third stationary point is $\left(\boldsymbol{\alpha\sqrt{\frac{\varepsilon}{(1-\varepsilon)}}, \frac{\alpha^2}{1-\varepsilon} - \frac{\alpha}{\sqrt{\varepsilon\sqrt{1-\varepsilon}}}}\right)$

Now let us find the Jacobian of the the system. For that let us find all the partial derivatives.

$(f_1)'_x = \frac{(\alpha^2 + x^2)(2xy) - 2x^3y}{(\alpha^2 + x^2)^2} + 1 - 2x,$

$$(f_1)'_y = \frac{-x^2}{(\alpha^2 + x^2)}$$

$$(f_2)'_x = \frac{(\alpha^2 + x^2)(2xy) - 2x^3y}{(\alpha^2 + x^2)^2}$$

$$(f_2)'_y = -\varepsilon + \frac{x^2}{(\alpha^2 + x^2)}$$

$$J = \begin{bmatrix} \frac{(\alpha^2 + x^2)(2xy) - 2x^3y}{(\alpha^2 + x^2)^2} + 1 - 2x & \frac{-x^2}{(\alpha^2 + x^2)} \\ \frac{(\alpha^2 + x^2)(2xy) - 2x^3y}{(\alpha^2 + x^2)^2} & -\varepsilon + \frac{x^2}{(\alpha^2 + x^2)} \end{bmatrix}$$

For (0, 0) :

J = $\begin{pmatrix} 1 & 0 \\ 0 & -\varepsilon \end{pmatrix}$,

then the characteristic equation for the Eigen values is given by

$\lambda^2 - (TrA)\lambda + \Delta = 0$.Here $TrA = (1-\varepsilon)$, and $\Delta = -\varepsilon$.

That gives us the case that

$$(TrA)^2 - 4\Delta) > 0 \,.\, i.e. \, the\ roots\ are\ real.$$

Since,$(1-\varepsilon)^2 - 4(-\varepsilon) = 1 - 2\varepsilon + \varepsilon^2 + 4\varepsilon = \varepsilon^2 + 2\varepsilon + 1 = \mathbf{(1+\varepsilon)^2} > 0$

Also, since $\varepsilon > 0$, $\therefore\ \Delta < 0$ Then the Eigen Values have different signs and this stationary state is a ***saddle point***

For (1,0) :

J = $\begin{bmatrix} -1 & -\frac{1}{\alpha^2+1} \\ 0 & -\varepsilon + \frac{1}{\alpha^2+1} \end{bmatrix}$,

thenthe Characteristic equation is given by

$\lambda^2 - (TrA)\lambda + \Delta = 0$. Here$TrA = \left(-1 - \varepsilon + \frac{1}{\alpha^2+1}\right),\ and\ \Delta = \varepsilon - \frac{1}{\alpha^2+1}$

That gives us the cases that

1) $(TrA)^2 - 4\Delta > 0 \,.\, i.e. \, the\ roots\ are\ real.$
2) $(TrA)^2 - 4\Delta < 0 \,.\, i.e. \, the\ roots\ are\ complex.$

Since, $\left(-1 - \varepsilon + \frac{1}{\alpha^2+1}\right)^2 = \left(\frac{(-\alpha^2 - 1 - \varepsilon\alpha^2 - \varepsilon + 1)}{\alpha^2+1}\right)^2 = \left(\frac{-\left((1+\varepsilon)\alpha^2 + \varepsilon\right)}{\alpha^2+1}\right)^2 = \left(\frac{\left((1+\varepsilon)\alpha^2+\varepsilon\right)^2}{(\alpha^2+1)^2}\right) \therefore\ (TrA)^2 - 4\Delta = \left(\frac{\left((1+\varepsilon)\alpha^2+\varepsilon\right)^2}{(\alpha^2+1)^2}\right) - 4\left(\varepsilon - \frac{1}{\alpha^2+1}\right)$. For fixed values of the parameter

Case1: $\boldsymbol{\varepsilon = 2}$, $\boldsymbol{\alpha = 1}$

$$(TrA)^2 - 4\Delta > 0\ also\ \Delta = 1.5 > 0\ and\ TrA = -0.5 < 0$$

∴The Eigen values have negative signs and the stationary state is a ***stable node.***

Case 2: But for $\boldsymbol{\varepsilon = 3}$, $\boldsymbol{\alpha = 2}$

$$(TrA)^2 - 4\Delta < 0\ also\ \Delta > 0\ and\ TrA < 0$$

∴ there will be an ***asymptotic stability***

And the stability is accquired for this particular parameter values. Other than this parameter values system is unstable for the parameter values near $\boldsymbol{\varepsilon = 3}$, $\boldsymbol{\alpha = 2}$ **.**

TO FIND THE LINEARISED SOLUTION OF THE SYSTEM

For the Jacobian above the Characteristic equation is given by

$$(-1-\lambda)\left(-\varepsilon + \frac{1}{\alpha^2+1} - \lambda\right) = 0 \implies \lambda = -1\ and\ \lambda = -\varepsilon + \frac{1}{\alpha^2+1}.$$

Therefore the ***Eigen Values*** are $\lambda = -1\ and\ \lambda = -\varepsilon + \frac{1}{\alpha^2+1}$ Let us find the corresponding Eigen Vectors. For that consider the equation,

$$\begin{bmatrix} -1-\lambda & -\frac{1}{\alpha^2+1} \\ 0 & -\varepsilon + \frac{1}{\alpha^2+1} - \lambda \end{bmatrix} \begin{bmatrix} h_1^1 \\ h_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To find the Eigen Vector for $\boldsymbol{\lambda = -1}$

$$\begin{bmatrix} -1+1 & -\frac{1}{\alpha^2+1} \\ 0 & -\varepsilon + \frac{1}{\alpha^2+1} + 1 \end{bmatrix} \begin{bmatrix} h_1^1 \\ h_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -\frac{1}{\alpha^2+1} \\ 0 & -\varepsilon + \frac{1}{\alpha^2+1} + 1 \end{bmatrix} \begin{bmatrix} h_1^1 \\ h_1^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \xRightarrow{(since\ the\ rank\ of\ co-eff.\ matrix\ is\ 1)} h_1^2 = 0\ and\ h_1^1 = k \in \mathbb{R}$$

$$\therefore\ h_1 = \begin{bmatrix} h_1^1 \\ h_1^2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$$

this is the eigen Vectorcorresponding to $\boldsymbol{\lambda = -1}$

To find the Eigen Vector for $\boldsymbol{\lambda = -\varepsilon + \frac{1}{\alpha^2+1}}$

$$\begin{bmatrix} -1+\boldsymbol{\varepsilon}-\dfrac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}} & -\dfrac{1}{\alpha^2+1} \\ 0 & -\varepsilon+\dfrac{1}{\alpha^2+1}+\boldsymbol{\varepsilon}-\dfrac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}} \end{bmatrix}\begin{bmatrix} h_2^1 \\ h_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1+\boldsymbol{\varepsilon}-\dfrac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}} & -\dfrac{1}{\alpha^2+1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} h_2^1 \\ h_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(-1+\boldsymbol{\varepsilon}-\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}}\right)h_2^1+\left(-\frac{1}{\alpha^2+1}\right)h_2^2 = 0$$

$$\left(-1+\boldsymbol{\varepsilon}-\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}}\right)h_2^1 = \left(\frac{1}{\alpha^2+1}\right)h_2^2$$

$$if\ \ h_2^1 = 1, then\ \ h_2^2 = \frac{\left(-1+\boldsymbol{\varepsilon}-\dfrac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}}\right)}{\left(\dfrac{1}{\alpha^2+1}\right)} = \alpha^2(\varepsilon-1)+\varepsilon-2$$

$$\therefore\ \ h_2 = \begin{bmatrix} h_2^1 \\ h_2^2 \end{bmatrix} = \begin{bmatrix} 1 \\ \alpha^2(\varepsilon-1)+\varepsilon-2 \end{bmatrix}$$

For fixed parameters $\underline{\boldsymbol{\varepsilon}=\mathbf{1}, \boldsymbol{\alpha}=\mathbf{1}}$ we have,

$$h_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

To determine the systems behaviour in the vicinity of the equilibrium u^*. The general solution has the form:

$$\delta(t) = \sum_{k=1}^{n} a_k h_k e^{\lambda_k t}$$

where and λ_{k}, h_k are the eigenvalues and the eigenvectors of $J\ (u^*)$ and coefficients are determined from the initial conditions

<u>To find linearized solution</u>

$$\begin{bmatrix} x(t)-1 \\ y(t)-0 \end{bmatrix} = \delta_1 \begin{bmatrix} k \\ 0 \end{bmatrix} e^{-1t} + \delta_2 \begin{bmatrix} 1 \\ \alpha^2(\varepsilon-1)+\varepsilon-2 \end{bmatrix} e^{(-\boldsymbol{\varepsilon}+\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}})t}$$

$$\begin{bmatrix} x(t)-1 \\ y(t)-0 \end{bmatrix} = \begin{bmatrix} \delta_1\, k\, e^{-1t} + \delta_2\, e^{(-\boldsymbol{\varepsilon}+\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}})t} \\ \delta_2\,(\alpha^2(\varepsilon-1)+\varepsilon-2)\, e^{(-\boldsymbol{\varepsilon}+\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}})t} \end{bmatrix}$$

$$x(t)-1 = \delta_1\, k\, e^{-1t} + \delta_2\, e^{(-\boldsymbol{\varepsilon}+\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}})t}$$

$$y(t) = \delta_2\,(\alpha^2(\varepsilon-1)+\varepsilon-2)\, e^{(-\boldsymbol{\varepsilon}+\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}})t}$$

<u>At t = 0</u>,

$$x(0)-1 = \delta_1\, k + \delta_2$$
$$y(0) = \delta_2\,(\alpha^2(\varepsilon-1)+\varepsilon-2)$$

Let us select the initial condition as $x(0) = 0.5, y(0) = 0.5$, we have

$$0.5-1 = \delta_1\, k + \delta_2 \implies -0.5 = \delta_1\, k + \delta_2 \qquad (A)$$
$$0.5 = \delta_2\,(\alpha^2(\varepsilon-1)+\varepsilon-2) \qquad (B)$$

Multiply (A) by$(\alpha^2(\varepsilon-1)+\varepsilon-2)$ and substract (B) from (A). and Substitute this value of $\delta_1$in (A) we get we get

$\delta_1 = \frac{-0.5(\alpha^2(\varepsilon-1)+\varepsilon-3)}{(\alpha^2(\varepsilon-1)+\varepsilon-2)k}$, $\delta_2 = -\frac{1}{2}$

Therefore the linearised solution is given by,

$$\boldsymbol{x(t)} = \mathbf{1} + \left(\frac{-0.5(\alpha^2(\varepsilon-1)+\varepsilon-3)}{(\alpha^2(\varepsilon-1)+\varepsilon-2)k}\right)\boldsymbol{k\, e^{-1t}} + \left(-\frac{1}{2}\right)\boldsymbol{e}^{(-\boldsymbol{\varepsilon}+\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}})\boldsymbol{t}}$$

$$\boldsymbol{y(t)} = \left(-\frac{1}{2}\right)(\boldsymbol{\alpha^2(\varepsilon-1)+\varepsilon-2})\, \boldsymbol{e}^{(-\boldsymbol{\varepsilon}+\frac{\mathbf{1}}{\boldsymbol{\alpha^2}+\mathbf{1}})\boldsymbol{t}}$$

For fixed parameters $\underline{\boldsymbol{\varepsilon}=\mathbf{1}, \boldsymbol{\alpha}=\mathbf{1}}$ we have,

$\boldsymbol{x(t)} = \mathbf{1} - \boldsymbol{e^{-t}} - \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{e}^{-\frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{t}}$and $\boldsymbol{y(t)} = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{e}^{-\mathbf{1}/\mathbf{2}\boldsymbol{t}}$

<u>For</u> $\left(\boldsymbol{\alpha}\sqrt{\frac{\boldsymbol{\varepsilon}}{(\mathbf{1}-\boldsymbol{\varepsilon})}}, \frac{\boldsymbol{\alpha^2}}{\mathbf{1}-\boldsymbol{\varepsilon}} - \frac{\boldsymbol{\alpha}}{\sqrt{\boldsymbol{\varepsilon(1-\varepsilon)}}}\right) \because x,y \geq 0\ and\ \alpha \neq 0 \implies \alpha > 0\ and\ 0 < \varepsilon < 1.$

Since $\frac{x^2}{(\alpha^2+x^2)} = \varepsilon$ the Jacobian simplifies to

$$J = \begin{bmatrix} 2y\varepsilon + 2y\varepsilon^2 + x - 2x^2 & -\varepsilon \\ \frac{2y}{x}(\varepsilon - \varepsilon^2) & 0 \end{bmatrix}$$

$$J_{\left(\alpha\sqrt{\frac{\varepsilon}{(1-\varepsilon)}}, \frac{\alpha^2}{1-\varepsilon} - \frac{\alpha}{\sqrt{\varepsilon\sqrt{1-\varepsilon}}}\right)}$$

$$= \begin{bmatrix} 2\left(\frac{\boldsymbol{\alpha^2}}{\boldsymbol{1-\varepsilon}} - \frac{\boldsymbol{\alpha}}{\sqrt{\boldsymbol{\varepsilon(1-\varepsilon)}}}\right)\varepsilon + 2\,(\frac{\boldsymbol{\alpha^2}}{\boldsymbol{1-\varepsilon}} - \frac{\boldsymbol{\alpha}}{\sqrt{\boldsymbol{\varepsilon(1-\varepsilon)}}})\,\varepsilon^2 + \boldsymbol{\alpha}\sqrt{\frac{\boldsymbol{\varepsilon}}{\boldsymbol{(1-\varepsilon)}}} - (2\boldsymbol{\alpha}\sqrt{\frac{\boldsymbol{\varepsilon}}{\boldsymbol{(1-\varepsilon)}}})^2 & -\varepsilon \\ \dfrac{2\left(\frac{\boldsymbol{\alpha^2}}{\boldsymbol{1-\varepsilon}} - \frac{\boldsymbol{\alpha}}{\sqrt{\boldsymbol{\varepsilon(1-\varepsilon)}}}\right)}{\boldsymbol{\alpha}\sqrt{\frac{\boldsymbol{\varepsilon}}{\boldsymbol{(1-\varepsilon)}}}}(\varepsilon - \varepsilon^2) & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} -2\alpha\varepsilon\sqrt{\frac{\varepsilon}{(1-\varepsilon)}} - \frac{2\alpha^2\varepsilon}{(1-\varepsilon)} + \frac{2\alpha^2\varepsilon^2}{\sqrt{1-\varepsilon}} & -\varepsilon \\ 2\alpha\sqrt{\varepsilon(1-\varepsilon)} - 1 + \varepsilon & 0 \end{bmatrix}$$

Here,

$$TrA = -2\alpha\varepsilon\sqrt{\frac{\varepsilon}{(1-\varepsilon)}} - \frac{2\alpha^2\varepsilon}{(1-\varepsilon)} + \frac{2\alpha^2\varepsilon^2}{\sqrt{1-\varepsilon}}, \quad \Delta = -\varepsilon(2\alpha\sqrt{\varepsilon(1-\varepsilon)} - 1 + \varepsilon)$$

To find the values of the parameters ε *and* α to check the Stability of the system.

For <u>$\boldsymbol{\varepsilon = 0.5}, \boldsymbol{\alpha = 1}$</u>

$$TrA = -2.2929 < 0, \quad \Delta = -.02500 < 0 \ \ and\ (TrA)^2 - 4\Delta = 8.2574 > 0$$

The Eigen values have negative signs and the stationary state is an ***unstable.***

For <u>$\boldsymbol{\varepsilon = 0.5}, \boldsymbol{\alpha = 2}$</u>

$$TrA = -7.1716 < 0, \quad \Delta = -1.7500 < 0 \ \ and\ (TrA)^2 - 4\Delta = 58.4318 > 0$$

The Eigen values have negative signs and the stationary state is an ***unstable.***

So the system is ***unstable*** for this stationary point for any value of $\alpha > 0$ *and* $0 < \varepsilon < 1$

For <u>$\boldsymbol{\varepsilon = 0.5}, \ \boldsymbol{\alpha = 2}$</u> the point is

$$\left(\boldsymbol{\alpha}\sqrt{\frac{\boldsymbol{\varepsilon}}{\boldsymbol{(1-\varepsilon)}}}, \quad \frac{\boldsymbol{\alpha^2}}{\boldsymbol{1-\varepsilon}} - \frac{\boldsymbol{\alpha}}{\sqrt{\boldsymbol{\varepsilon(1-\varepsilon)}}}\right) = (\mathbf{1}, \mathbf{1})$$

IV. Numerical Solution using RK 6th order method for a system of D.E. and Multistep method

Here we investigate the given Model using numerical methods like Runge-Kutta 6th order Method and Multistep Method etc. We use of MATLAB to write the codes for all the methods.

Before writing the codes, we have derived the following formulae for the RK6. We find the expressions for u and then represent it in terms of $k_i's$

$$\widetilde{u_1} = u + h[a_{11}\ f(\widetilde{u_1})]$$
$$\widetilde{u_1} = u$$
$$\widetilde{u_2} = u + h[a_{11}\ f(\widetilde{u_1}) + a_{12}\ f(\widetilde{u_2})]$$
$$\widetilde{u_2} = u + \frac{h}{5}\ f(\widetilde{u_1})$$
$$\widetilde{u_3} = u + h[a_{11}\ f(\widetilde{u_1}) + a_{12}\ f(\widetilde{u_2}) + a_{13}\ f(\widetilde{u_3})]$$
$$\widetilde{u_3} = u + h\left[\frac{3}{40} f(\widetilde{u_1}) + \frac{9}{40} f(\widetilde{u_2})\right]$$
$$\widetilde{u_4} = u + h\left[\frac{3}{10} f(\widetilde{u_1}) + \frac{9}{10} f(\widetilde{u_2}) + \frac{12}{10} f(\widetilde{u_3})\right]$$
$$\widetilde{u_5} = u + h\left[\frac{3}{40} f(\widetilde{u_1}) + \frac{27}{40} f(\widetilde{u_2}) - \frac{24}{40} f(\widetilde{u_3}) + \frac{30}{40} f(\widetilde{u_4})\right]$$
$$\widetilde{u_6} = u + h\left[\frac{107}{162} f(\widetilde{u_1}) + \frac{5}{2} f(\widetilde{u_2}) - \frac{140}{27} f(\widetilde{u_3}) + \frac{35}{9} f(\widetilde{u_4}) - \frac{70}{89} f(\widetilde{u_5})\right]$$

<u>**To find *Ki's***</u>

$\boldsymbol{k1 = hf(\widetilde{u_1})}$,

$k2 = hf(\widetilde{u_2})$,

$$k2 = hf\left(u + \frac{hf(\widetilde{u_1})}{5}\right)$$

$$\boldsymbol{k2 = hf\left(u + \frac{k1}{5}\right)}$$

$$k3 = hf(\widetilde{u_3})$$

$$k3 = hf\left(u + \frac{3}{40}hf(\widetilde{u_1}) + \frac{9}{40}hf(\widetilde{u_2})\right)$$

$$\boldsymbol{k3 = hf\left(u + \frac{3}{40}k1 + \frac{9}{40}k2\right)}$$

$$k4 = hf(\widetilde{u_4})$$

$$k4 = hf\left(u + \frac{3}{10}hf(\widetilde{u_1}) - \frac{9}{10}hf(\widetilde{u_2}) + \frac{12}{10}hf(\widetilde{u_3})\right)$$

$$\boldsymbol{k4 = hf\left(u + \frac{3}{10}k1 - \frac{9}{10}k2 + \frac{12}{10}k3\right)}$$

$$k5 = hf(\widetilde{u_5})$$

$$k5 = hf\left(u + \frac{3}{40}hf(\widetilde{u_1}) + \frac{27}{40}hf(\widetilde{u_2}) - \frac{24}{40}hf(\widetilde{u_3}) + \frac{30}{40}hf(\widetilde{u_4})\right)$$

$$\boldsymbol{k5 = hf\left(u + \frac{3}{40}k1 + \frac{27}{40}k2 - \frac{24}{40}k3 + \frac{30}{40}k4\right)}$$

$$k6 = hf(\widetilde{u_6})$$

$$k6 = hf\left(u + \frac{107}{162}hf(\widetilde{u_1}) + \frac{5}{2}hf(\widetilde{u_2}) - \frac{140}{27}hf(\widetilde{u_3}) + \frac{35}{9}hf(\widetilde{u_4}) - \frac{70}{81}hf(\widetilde{u_5})\right)$$

$$k6 = hf\left(u + \frac{107}{162}k1 + \frac{5}{2}k2 - \frac{140}{27}k3 + \frac{35}{9}k4 - \frac{70}{81}k5\right)$$

$$\boldsymbol{k6 = hf\left(u + \frac{107}{162}k1 + \frac{5}{2}k2 - \frac{140}{27}k3 + \frac{35}{9}k4 - \frac{70}{81}k5\right)}$$

a. **Exploring the Numerical solution of the given system for different values of parameter**

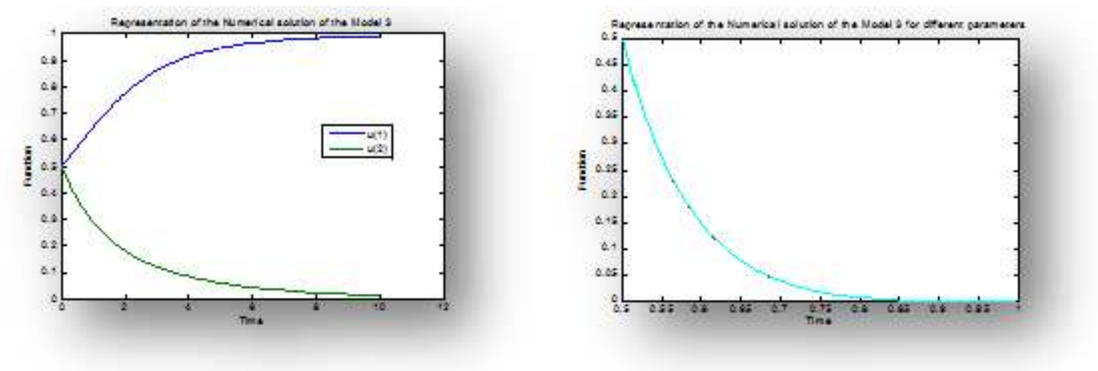

***Figure** 1: Representation of the Numerical Solution of Given Bacterial Model A(Model 3)*

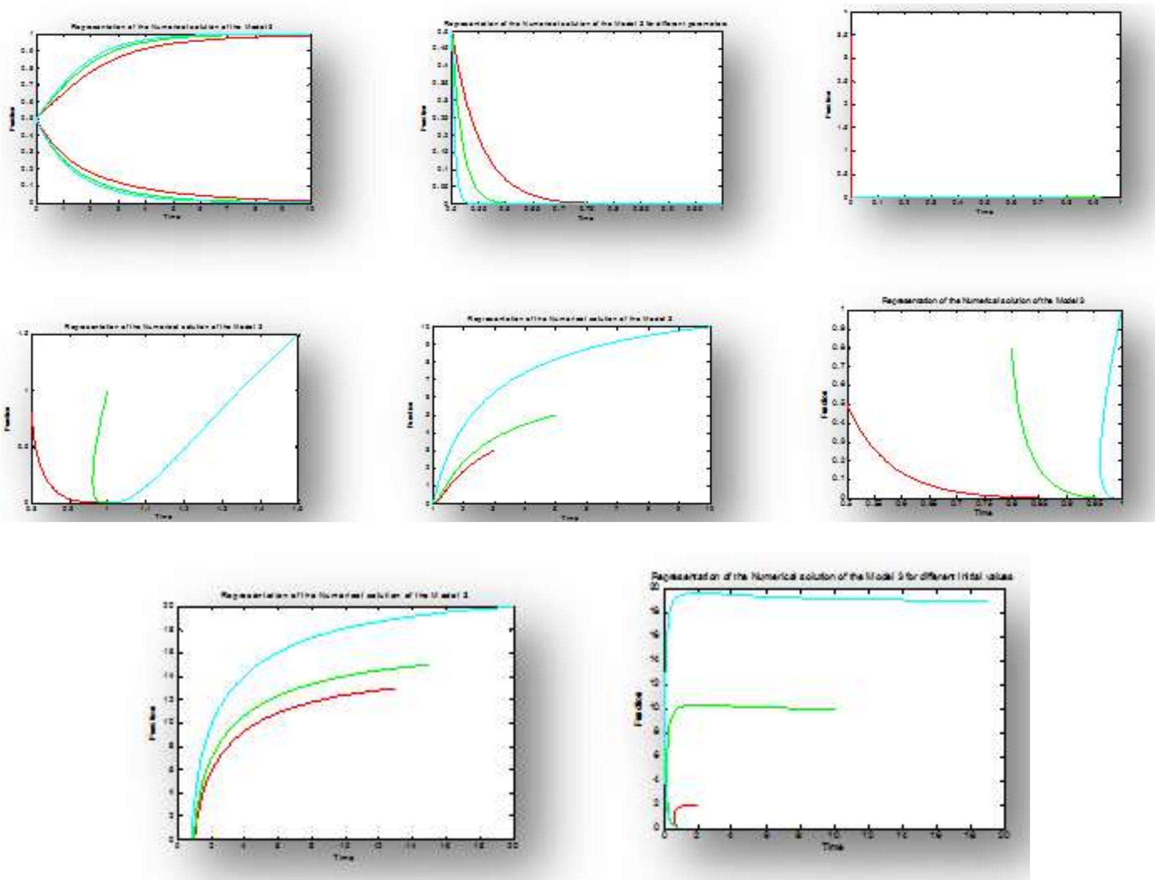

Figure 2: Representation of the Numerical Solution of Given Bacterial Model A, for different parameters (Model 3)

The system is stable if the parameters take the correct values which satisfy the conditions for the system. For big values of parameters, a and e the system becomes unstable. If the parameters take very big negative values the system looks like *figure 2*. For different parameters the stability of the system changes it becomes unstable if we select large values of the parameters.

Fix the parameters and change the initial condition: For some fixed value of parameter(s), compare the solution obtained from the linear approximation to the numerical solution starting from the same initial condition in the vicinity of an equilibrium point.

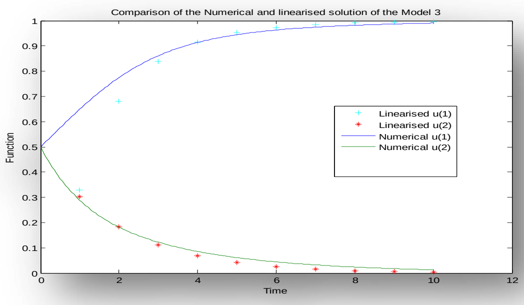

Figure 3: Representation of comparison of the Exact and Numerical Solution of Given Bacterial Model A(Model 3

b. **MATLAB codes for the two new methods** (one one-step and one multistep method) and use these methods to obtain numerical solutions for the given model.

Code for rk6

```
function unew = rk6(u, h, flow)
k1 = h.*flow(u);
k2 = h.*flow(u + k1/5);
```

k3 = h.*flow(u + 3*k1/40 + 9*k2/40);
k4 = h.*flow(u + 3*k1/10 - 9*k2/10 + 12*k3/10);
k5 = h.*flow(u + 3*k1/40 + 27*k2/40 -3*k3/5 + 3*k4/4);
k6 = h.*flow(u + 107*k1/162 + 5*k2/2 -140*k3/27 + 35*k4/9 - 70*k5/81);
unew = u + (8*k1/81 + 25*k3/63 + 25*k4/108 + 25*k5/81 - k6/28);

Code for Numerical solution of the given system using rk6

```
>> global a e
>> a = 1
>> e = 0.8
>> [t, u] = ODEsolver([0.5 0.5], 0.05, 10, @Model3, @rk6)
```

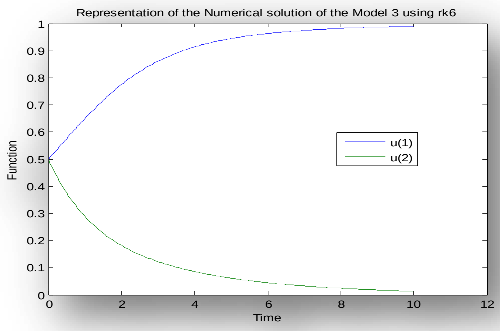

Figure 4: Representation of Numerical Solution of Given Bacterial Model A by RK6 method (Model 3)

code for multistep

```
clear all
close all
global a e
T = 10; N = 200; h = T/N;
a = 1; e = 1;
u(1,1) = 0.5; u(1,2) = 0.5; t(1) = 0;
for i = 2:5
    u(i,:) = rk4(u(i-1,:), h, @Model3);
    f(i,:) = Model3(u(i,:));
    t(i) = t(i-1) + h;
end
for i = 6:N
u1    =    u(i-1,:)+h/720.*(1901*f(i-1,:)-2774*f(i-2,:)+2616*f(i-3,:)-
1274*f(i-4,:)+251*f(i-5,:));
ff = Model3(u1);
u1   =   u(i-1,:)+h/720.*(251*ff+646*f(i-1,:)-264*f(i-2,:)+106*f(i-3,:)-
19*f(i-4,:));
u(i,:) = u1;
f(i,:) = Model3(u(i,:));
t(i) = t(i-1)+h;
end
plot(t, u(:,1),'g')
hold on
plot(t, u(:,2),'r')
xlabel('Time'); ylabel('Function of Model 3)');
title('Numerical solution of Model 3 using rk6');
```

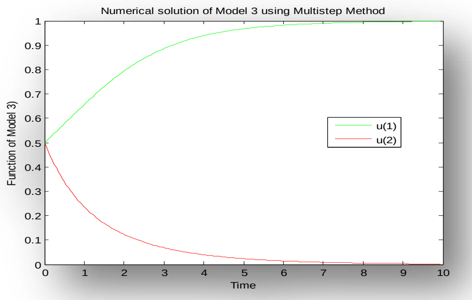

Figure 5: Representation of Numerical Solution of Given Bacterial Model A by Multi-step method (Model 3)

c. **step-size-error diagrams for the two methods and determine the order of the methods.**

```
clear all
close all
global a e
format long
T = 10; h=0.05; N = T/h;
a = 1; e = 1;
u0(1) = 0.1; u0(2) = 0.1; t(1) = 0;
h_exact = h/32;
h_approx = [h/4 h/2 h 2*h 4*h 8*h 16*h];
figure(1); clf;
[t, uexact] = ODEsolver(u0, h_exact, T, @Model3, @rk6);
for i = 1:length(h_approx)
    clear tuerr1
    K = h_approx(i)/h_exact;
    [t, u] = ODEsolver(u0, h_approx(i), T, @Model3, @rk6);
    err(1,:) = [0 0];
for j = 2:length(u)
        err1(j,:) = (abs(uexact((j-1)*K + 1,: ) - u(j,:)));
end
    glob_err(i,1) = max(err1(:,1));
    glob_err(i,2) = max(err1(:,2));
end
plot(log(h_approx),log(glob_err(:,1)),'r*-');
hold on
plot(log(h_approx),log(glob_err(:,2)),'g*-');
xlabel('log(h)'); ylabel('log(global err)');
grid on;
title('Stepsize error diagram for runge-kutta method');
```

Figure 6: Representation step-size error diagram of Given Bacterial Model A by Runge-Kutta method (Model 3)

The order of the above method is 5 as seen from the diagram, as the slope of the lines is 5

Code for multistep method

```
function [t,u] = multistep1(u0,T,h,flow)
```

```
N = floor(T/h);
t = zeros(N+1,1);
u(1,1) = u0(1);
u(1,2) = u0(2);
t(1) = 0;
f = flow(u);
for i = 2:5
    u(i,:) = rk4(u(i-1,:), h, flow);
    f(i,:) = flow(u(i,:));
    t(i) = t(i-1) + h;
end
for i = 6:N+1
    u1    =   u(i-1,:)+h/720.*(1901*f(i-1,:)-2774*f(i-2,:)+2616*f(i-3,:)-
1274*f(i-4,:)+251*f(i-5,:));
    ff = flow(u1);
    u1    =    u(i-1,:)+h/720.*(251*ff+646*f(i-1,:)-264*f(i-2,:)+106*f(i-
3,:)-19*f(i-4,:));
      u(i,:) = u1;
      f(i,:) = flow(u(i,:));
      t(i) = t(i-1)+h;
   end
```

Code for Step-Size Error of Predictor-Corrector Method

```
clear all
close all
global a e
T = 10; h=0.01; N = T/h;
a = 1; e = 1;
u0(1) = 0.1; u0(2) = 0.1; t(1) = 0;
h_exact = h/32;
h_approx = [h/4 h/2 h 2*h 4*h 8*h 16*h];
[t,u_exact] = multistep1(u0, T, h_exact, @Model3);
for i = 1:length(h_approx)
    clear tuerr1
    K = h_approx(i)/h_exact;
    [t,u] = multistep1(u0, T, h_approx(i), @Model3);
    err(1,:) = [0 0];
for j = 2:length(u)
        err1(j,:) = (abs(u_exact((j-1)*K + 1,: ) - u(j,:)));
end
    glob_err(i,1) = max(err1(:,1));
    glob_err(i,2) = max(err1(:,2));
end
plot(log(h_approx),log(glob_err(:,1)),'r*-');
hold on
plot(log(h_approx),log(glob_err(:,2)),'g*-');
xlabel('log(h)'); ylabel('log(global err)');
grid on;
title('Step-size error diagram for predictor-corrector method');
```

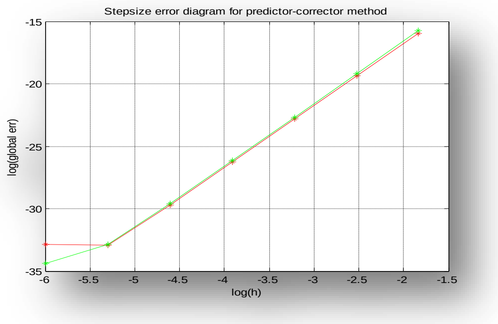

Figure 7: Representation step-size error diagram of Given Bacterial Model A by Multi-step method (Model 3)

The order of the above method is 5 as seen from the diagram, as the slope of the lines is 5

V. Conclusions

The Runge-Kutta 6th order method was developed to simulate model numerically and predictor-corrector methods is used for comparison. From the above discussion it is clear that the linear stability analysis helps in determining the performance of the system and also the step size error diagram helps in determining the order of the method. In future we will investigate other mathematical models for Bacterial growth to determine the stability of the model. The MATLAB codes given in the paper will help the new researchers to initiate skills in simulation of the models.

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