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OscillationsofThird Order Neutral DelayDifferential Equations

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Abstract: Sufficient conditions for oscillations of third order non-linear neutral delay differentialequations of the form

$$\frac{d}{dt}\left\{r(t)\frac{d^2}{dt^2}\left(m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)\right)\right\}+f(t)y(t-\sigma)=0,\quad t\geq t_0$$

are obtained where, $r(t),m(t)$ are positive real valued continuous functions $f(t)\geq 0$, and α is the ratio of odd positive integers and $n$is an integer.
Key words: Oscillation, Third order, Neutral Differential equation.

I. INTRODUCTION

In this paper we consider the non-linear neutral delay differential equation

$$\frac{d}{dt}\left\{r(t)\frac{d^2}{dt^2}\left(m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)\right)\right\}+f(t)y(t-\sigma)=0,\quad t\geq t_0 \tag{1}$$

where $r(t)\in C([t_0,\infty),(0,\infty))$, $f(t)\in C([t_0,\infty),[0,\infty))$

Corresponding equation in the absence of neutral term is given by

$$\frac{d}{dt}\left\{r(t)\frac{d^2}{dt^2}\{m(t)y(t)\}\right\}+f(t)y(t-\sigma)=0 \tag{2}$$

which is a delay differential equation and further if we take $m(t)=1,\sigma=0$ in equation (2) we get

$$\frac{d}{dt}\left\{r(t)\frac{d^2}{dt^2}\{y(t)\}\right\}+f(t)y(t)=0 \tag{3}$$

The study of behavior of solutions of differential equation (2) has been a subject of interest for several researchers. We mention the works of [4, 5and 6].Oscillatory behavior of delay differential equations is extensively studied by several authors [1, 2, 3,7,10,11,12,13,14, 15 and 16].

Now we see some special case of equation (1).

When
$r(t)\equiv 1$equation (1) is reduced to

$$\frac{d^3}{dt^3}\left\{m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)\right\}+f(t)y(t-\sigma)=0 \tag{4}$$

and to

$$\frac{d^3}{dt^3}\left\{m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)\right\}+f(t)y(t)=0 \text{ if } \quad \sigma=0 \tag{5}$$

and we note that, when $m(t)=1$, this equation further becomes to the equation

$$\frac{d^3}{dt^3}\left\{y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)\right\}+f(t)y(t)=0 \tag{6}$$

Recently there has been an increasing interest in the study of the oscillation of differential equations e.g. papers [18]-[22]. In particular, differential equations of the form (1) and for special cases when $r(t) \equiv 1$, is a subject of intensive research.

The oscillation for equation (6) has been discussed by many authors.

CuimeiJiang ,Ying Jiang and Tongxing Li [8],studied the Asymptotic behavior of Third order differential equations with nonpositive neutral coefficients and distributed deviating arguments of the form

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + \int_{c}^{d} q(t,\xi) f\left[x(\sigma(t,\xi))\right] d\xi = 0 \qquad (7)$$

andMustafa FahriAktas ,DevrimCakmakand AydinTiryaki [17] have considered the Third order nonlinear functional differential equations of the form

$$\left(r_2(t)\left(r_1(t)y'\right)'\right)' + p(t)y' + q(t) f\left(y(g(t))\right) = 0 \qquad (8)$$

V. Ganesan ,M. Sathish Kumar [9] have considered the Third order nonlinear neutral differential equation with neutral terms of the form

$$\left[r(t)\left[\left(x(t) + \sum_{i=1}^{n} p_i(t) x(\eta_i(t)) \right)''\right]^{\gamma}\right]' + q(t) x^{\gamma}(\sigma(t)) = 0 \qquad (9)$$

The present work is motivated by [19] where the Authors, P. V. H. S Sai Kumar and K. V. V SeshagiriRaohave considered oscillations of third order linear neutral delay differential equation of the form

$$\frac{d}{dt}\left\{ r_1(t) \frac{d^2}{dt^2}\left(m(t)y(t) + \frac{r(t)}{r(t-\tau)} y^{\alpha}(t-\tau) \right)\right\} + f(t)y(t-\sigma) = 0 \;; \qquad t \ge t_0$$

In this paper we establish the conditions for the oscillation of solutions of equation (1) by Ricccati Technique using the condition.

$\int_{t_0}^{t} \frac{1}{r(t)} dt = \infty$ as $t \to \infty$.

By a solution of equation (1) we mean a function $y(t) \in C([T_y, \infty))$ where $T_y \ge t_0$ which satisfies(1) on $[T_y, \infty)$. We consider only those solutions of y(t) of (1) which satisfy $Sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$ and assume that (1) possesses such solutions.

A solution of equation (1) is called oscillatory if it has arbitrary large zeros on $[T_y, \infty)$; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory it all its solutions oscillate .Unless otherwise stated, when we write a functional inequality, it will be assumed to hold for sufficiently large t in our subsequent discussion.

II. MAIN RESULTS

We need the following in our discussion

(H_1): $r(t), m(t) \in C([t_0, \infty), R)$;

(H_2): $f(t)$, $p_i(t) = \frac{1}{t_i(t_i - 1)}$ are continuously differentiable on $[t_0, \infty)$.

(H_3): $0 < \alpha \le 1$, and α is the ratio of odd positive integers.

(H_4): τ and σ are positive constants

(H_5) $f(t) > 0$, $0 < \sum_{i=1}^{n} p_i(t) < \infty$ i.e. $0 < \sum_{i=1}^{n} \frac{1}{t_i(t_i - 1)} < \infty$ for $i = 1, 2 \ldots\ldots \infty$.

We set

$$z(t)=m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau) \tag{10}$$

And $R(t)=\int_{t_0}^{t}\frac{1}{r(t)}dt=\infty$ as $t\to\infty$ (11)

We have the following Lemmas

Lemma 2.1: $If\ \ a\ and\ \ b$ arepositive,then $a^{\alpha}b^{1-\alpha}\le \alpha a+(1-\alpha)b \quad for \quad 0<\alpha\le 1,$

where equality holds if and only if $a=b.$

Lemma 2.2: Let $\alpha\ge 1$, be a ratio of odd positive integers. Then

$$Bu-Au^{\frac{\alpha+1}{\alpha}}\le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \qquad A,B>0, \tag{12}$$

Now we present the main theorem.

Theorem 2.1: Assume $(H_1)-(H_5)$ and (11) hold. If $\alpha\ge 1$ and there exists a positive non decreasing function $\rho\in C^{'}([t_0,\infty),R)$ such that

$$\underset{t\to\infty}{Lim\sup}\int_{t_2}^{t}\left[\left\{\frac{\rho(s)f(s)}{m(s-\sigma)}\left[2k\left(1-[\alpha+(1+\alpha)\frac{1}{k}]\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\right)\right]\right\}-\frac{(\rho'(s))^2 r(s)}{\rho(s)}.\right]ds=\infty \tag{13}$$

for some $k\in(0,1)$, then every solution of equation (1) is oscillatory.

Proof. Suppose that there exists a nonoscillatory solution. Let $y(t)$ be such nonoscillatory solution. Without loss

of Generality we can assume that there exists $T>t_0$ such that $y(t)>0$ for $t\ge T$.

We observe that

$$z(t)>0,\quad z'(t)>0,\quad z''(t)>0,\big(r(t)z'''(t)\big)\le 0;\ \text{for}\ t\ge t_1\ge T>t_0$$

Thus $\quad z(t)\ge a(t)z''(t)R(t)\ ,\ t\ge t_1$

$r(t)z''(t)$ is decreasing. Hence $\dfrac{z(t)}{R(t)}$ is decreasing for $t\ge t_1$.

Since $z(t)$ is positive increasing function, there exists a constant $k>0$ such that

$$z(t)\ge 2k>0 \tag{14}$$

From the definition of $z(t)$ we have

$$z(t)=m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)$$

$$m(t)y(t)=z(t)-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)$$

$$y(t)\ge\frac{1}{m(t)}\left[z(t)-\sum_{i=1}^{n}\frac{1}{t(t-1)}z^{\alpha}(t)\right]\ , \qquad t-\tau\le t$$

$$y(t)\ge\frac{1}{m(t)}\left[z(t)-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\big(\alpha\, z(t)+(1-\alpha)\big)\right]$$

$$y(t)\ge\frac{1}{m(t)}\left[z(t)-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha\, z(t)-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right]$$

$$y(t)=\frac{1}{m(t)}\left[z(t)\left(1-\sum_{i=1}^{n}\frac{\alpha}{t_i(t_i-1)}\right)-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right] \tag{15}$$

We have used the inequality with b=1.

Using $p(t)=\max_{i=1,\ldots\ldots.m} p_i(t)$

Also we have
$$y(t)\geq\frac{1}{m(t)}\left[z(t)\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha-\frac{1}{z(t)}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right] \tag{16}$$

Substituting (14) into (16) we get

$$y(t)\geq\frac{1}{m(t)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha-\frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right] \tag{17}$$

From equation (1) we see that

$$\frac{d}{dt}\left\{r(t)\frac{d^2}{dt^2}\left(m(t)y(t)+\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}y^{\alpha}(t-\tau)\right)\right\}=-f(t)y(t-\sigma)$$

But

$$y(t)\geq\frac{1}{m(t)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha-\frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right]$$

$$y(t-\sigma)\geq\frac{1}{m(t-\sigma)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha-\frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right] \tag{18}$$

Define

$$\omega(t)=\rho(t)\frac{r(t)z''(t)}{z'(t)};\quad t\geq t_1 \tag{19}$$

$$\omega'(t)=\rho'(t)\frac{r(t)z''(t)}{z'(t)}+\rho(t)\left[\frac{r(t)z''(t)}{z'(t)}\right]'$$

$$\omega'(t)=\rho'(t)\frac{r(t)(z''(t))}{z'(t)}+\rho(t)\left[\frac{z'(t)(r(t)z''(t))'-(r(t)z''(t))z''(t)}{(z'(t))^2}\right]$$

$$\omega'(t)=\rho'(t)\frac{r(t)(z''(t))}{z'(t)}+\rho(t)\left[\frac{z'(t)(r(t)z''(t))'}{z'(t)z'(t)}\right]-\rho(t)\left[\frac{(r(t)z''(t))z''(t)}{(z'(t))^2}\right]$$

$$\omega'(t)=\rho'(t)\frac{r(t)(z''(t))}{z'(t)}+\rho(t)\left[\frac{(r(t)z''(t))'}{z'(t)}\right]-\rho(t)\left[\frac{r(t)(z''(t))^2}{(z'(t))^2}\right]$$

From (19) we have,

$$\frac{\omega(t)}{\rho(t)}=\frac{r(t)(z''(t))}{z'(t)},$$

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha - \frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right]\right\}$$
$$-\rho(t)\left[\frac{r(t)(z''(t))^{2'}}{(z'(t))^2}\right] \tag{20}$$

Also

$$\frac{z''(t)}{z'(t)} = \frac{\omega(t)}{r(t)\rho(t)}$$

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha - \frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right]\right\}$$
$$-\rho(t)\left[r(t).\frac{\omega^2(t)}{r^2(t)\rho^2(t)}.\right]$$

$$\omega'(t) \le \frac{\rho'(t)}{\rho(t)}\omega(t) - \rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha - \frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right]\right\}$$
$$-\left[\frac{\omega^2(t)}{r(t)\rho(t)}.\right] \tag{21}$$

$$u = \omega(t)\ , \qquad A = \frac{1}{r(t)\rho(t)}\quad , \qquad B = \frac{\rho'(t)}{\rho(t)}$$

Using the Inequality

$$Bu - Au^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha}, \qquad A, B > 0,$$

$$\frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{1}{r(t)\rho(t)}\omega^2(t) \le \frac{1}{4}\frac{\left(\frac{\rho'(t)}{\rho(t)}\right)^2}{\frac{1}{r(t)\rho(t)}}$$

$$\frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{1}{r(t)\rho(t)}\omega^2(t) \le \frac{1}{4}\left(\frac{\rho'(t)}{\rho(t)}\right)^2.r(t)\rho(t)$$

$$\frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{1}{r(t)\rho(t)}\omega^2(t) \le \frac{1}{4}\frac{(\rho'(t))2}{\rho(t)}.r(t) \tag{22}$$

From (21)

$$\omega'(t) \le -\rho(t)f(t)\left\{\frac{1}{m(t-\sigma)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha - \frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right]\right\}$$
$$+\frac{\rho'(t)}{\rho(t)}\omega(t) \quad -\left[\frac{\omega^2(t)}{r(t)\rho(t)}.\right]$$

From (22)

$$\omega'(t) \le -G(t) + \frac{1}{4}\frac{(\rho'(t))2}{\rho(t)}.r(t)$$

where

$$G(t) = -\rho(t) f(t)\left\{\frac{1}{m(t-\sigma)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha - \frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right]\right\}$$

Hence there exists $t_2 > t_1$ such that

$$\int_{t_2}^{t}.\left(\rho(s) f(s)\left\{\frac{1}{m(s-\sigma)}\left[2k\left(1-\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}\alpha - \frac{1}{k}.\sum_{i=1}^{n}\frac{1}{t_i(t_i-1)}(1-\alpha)\right)\right]\right\} - \frac{1}{4}\frac{(\rho'(s))^2 r(s)}{\rho(s)}\right)ds \le \omega(t_2)$$

which contradict (13).Hence the proof is complete.

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