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F - Contraction on Common Fixed Point Theorem in Complete *b* - Metric Spaces

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Abstract: In this paper, using the concept of F - contraction, we first establish a unique common fixed point theorem for two self mapping on complete b-metric spaces.
The results extend and generalize some results in the literature.
Keywords: Complete b- metric space, common fixed points, F-contractions.

I. INTRODUCTION

In 1922, Polish mathematician Banach [5] proved a very important result regarding a contraction mapping, known as the Banach contraction principle. The Banach contraction principle is a popular tool in solving existence problems in many branches of mathematics, this result has been extended in many directions.

In 2012, Wardowski [20,21] introduced a new type of contractions called F-contraction and proved a new fixed point theorem concerning F-contraction.

In 1989, Bakhtin[6] introduced the notion of b metric space ,which was formally defined by Czerwik[12] in 1993 .Fixed point theorem for various contractions in b metric spaces were discussed in [3,4,7,8,16]. There are many authors who have worked on the generalization of fixed point theorems in b metric spaces for example [9,14,17,19].

In this paper we will apply Hardy-Rogers-type F-contraction mapping for two self mappings on complete b metric space.

The aim of this paper is to establish some new common fixed point theorems and generalize some of the results in the literature on F-contractions.

II. PRELIMINARIES

We recall some basic known definitions and results which will be used in the sequel.

Definition 2.1 [6,12]. Let X be a non-empty set and $s \geq 1$ a real number. A function $d : X \times X \longrightarrow [0, \infty)$ is called a b-metric if the following conditions are satisfied, for every $x, y, x^* \in X$:

(B1) $d(x, y) = 0$ if and only if $x = y$;
(B2) $d(x, y) = d(y, x)$;
(B3) $d(x, y) \leq s\,[d(x, x^*) + d(x^*, y)]$.

In this case (X, d) is called a b-metric space with constant $s \geq 1$.

Example 2.2 [13].
There follow two other examples:
1. Let $X = [0, 2]$ and $d : X \times X \to [0,\infty)$ be defined by

$$d(x, y) = \begin{cases} (x-y)^2 & x, y \in [0,\ 1] \\ \frac{1}{x^2} - \frac{1}{y^2}, & x, y \in [1, 2] \\ |x-y|, & Otherwise \end{cases}$$

It can easily be seen that d is a b-metric on X and so, (X, d, s) is a b-metric space with $s = 2$.
2. Let $X = \{1, 2, 3, 4\}$ and define $d : X \times X \to [0, \infty)$ as follows:
$d(n, n) = 0$, $n = 1, 2, 3, 4$;
$d(1, 2) = d(2, 1) = 2$;

$d(2, 3) = d(3, 2) = \frac{1}{2}$;
$d(1, 3) = d(3, 1) = 1$;
$d(1, 4) = d(4, 1) = \frac{3}{2}$
$d(2, 4) = d(4, 2) = d(3, 4) = d(4, 3) = 3$.
Then d is a b-metric with $s = 2$.

Lemma 2.3. [16]
If (X, d) is a b-metric space with constant $s \geq 1$, $x^*, y^* \in X$ and $(x_n)_{n\in N}$ is a convergent sequence in X with $\lim_{n\to\infty} x_n = x^*$ then
$\frac{1}{s} d(x^*, y^*) \leq \lim_{n\to\infty} \inf d(x_n, y^*) \leq \lim_{n\to\infty} \sup d(x_n, y^*) \leq sd(x^*, y^*)$.

Definition 2.4. [20, 21] A function $F : (0, \infty) \longrightarrow \mathbb{R}$ be a map satisfying the following conditions:
(F1) F is strictly increasing;
(F2) For each sequence $\{\propto_n\}_{n\in N}$ of positive numbers $\lim_{n\to\infty} \propto_n = 0$ if and only if $\lim_{n\to\infty} F(\propto_n) = -\infty$.
(F3) There exists $k \in (0, 1)$ such that $\lim_{\propto \to 0^+} \propto^k F(\propto) = 0$.
For a metric space (X, d), a mapping $T : X\longrightarrow X$ is said to be a Wardowski F –contraction if there exists $\tau > 0$ *such that* $d(Tx, Ty) > 0$ implies
$\tau + F(d(Tx, Ty)) \leq F(d(x, y))$,
For all $x, y \in X$.
In 2015 Cosentino et al.[10] attempted to apply Wardowski's method in the context of b metric space, by using the following additional assumption ,
(F4) Let $s \geq 1$, be a real number .For each sequence $\{\propto_n\}_{n\in N}$ of positive numbers such that $\tau + F(s \propto_n) \leq F(\propto_{n-1})$ for all $n\in N$ and some $\tau > 0$ *then*
$\tau + F(s^n \propto_n) \leq F(s^{n-1}\propto_{n-1})$ for all $n\in N$.

Definition 2.5. [16] Let (X, d) be a b-metric space with constant $s \geq 1$, and $T : X\longrightarrow X$ an operator. If there exist $\tau > 0$ and $F \in \mathcal{F}_{s,\tau}$ such that for all $x, y \in X$ the inequality $d(Tx, Ty) > 0$ Implies
(F) $\tau + F(s\, d(Tx, Ty)) \leq F(d(x, y))$.
Then T is called an *F* contraction.

Theorem 2.6 [16] If (X, d) is a complete b-metric space with constant $s \geq 1$ and $T : X \longrightarrow X$ is an F-contraction for some $F \in \mathcal{F}_{s,\tau}$ then T has a unique fixed point x^*. Furthermore, for any $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ is convergent and $\lim_{n\to\infty} x_n = x^*$.
Convergent sequences and Cauchy sequences in b metric spaces, etc are defined in the same way as in metric spaces.
Motivated by these ideas, here we define a new type of F - contraction of Hardy -Rogers type for two self mappings in b metric space and prove a unique common fixed point theorem.

III. MAIN RESULTS

Definition 3.1. Let (X, D, s) be a b metric space with constant $s \geq 1$. Let $a_1, a_2, a_3, a_4, a_5 \geqq 0$ real number and S and T are self mappings on X. If there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $d(Sx, Ty) > 0$ implies
$\tau + F(s\, d(Sx, Ty)) \leq F(a_1 d(x, y) + a_2 d(x, Sx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Sx)$, … (3.1)
Then T and S are called F - contraction of Hardy-Rogers-type.
Theorem 3.2. Let (X, D, s) be a complete b metric space with constant $s \geq 1$ and let T, S be are two self mappings on X. Assume that there exists $F \in F$ and $\tau > 0$ such that T and S are F - contraction of Hardy-Rogers-type i.e.
$\tau + F(s\, d(Sx, Ty)) \leq F(a_1 d(x, y) + a_2 d(x, Sx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Sx)$, for all $x, y \in X$ with $d(Sx, Ty) > 0$,
if either $a_1 + a_2 + a_3 + (s+1) a_4 < 1$, $a_3 \neq 1$, $a_5 \geq 0$
or $a_1 + a_2 + a_3 + (s+1) a_5 < 1$, $a_3 \neq 1$, $a_4 \geq 0$,
Then T and S have common fixed point. Moreover if $a_1 + a_4 + a_5 < s$ holds as well then the common fixed point of S and T is unique.
Proof:- Let $x_0 \in X$ be any point, which is arbitrary, and let,

$Sx_{2n} = x_{2n+1}$, $Tx_{2n+1} = x_{2n+2}$, for all n = 0, 1, 2, ...
Using the contractive condition (3.1) with $x_1 = Sx_0$, and $x_2 = Tx_1$, $d_n = (x_n, x_{n+1})$ when $n = 0$, we get, $\tau + F(sd(x_1, x_2)) = \tau + F(sd(Sx_0, Tx_1))$
$\leq F(a_1 d(x_0, x_1) + a_2 d(x_0, Sx_0) + a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + a_5 d(x_1, Sx_0)$
$\leq F(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + a_5 d(x_1, x_1)$
$\leq F(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 (s d(x_0, x_1) + s d(x_1, x_2))$
$\leq F((a_1 + a_2 + sa_4)\ d(x_0, x_1)) + d(x_1, x_2)(a_3 + sa_4))$
$\tau + F(sd_1) \leq F((a_1 + a_2 + sa_4) d_0 + d_1 (a_3 + sa_4))$
since F is strictly increasing, it follows that

$$sd_1 < (a_1 + a_2 + sa_4) d_0 + d_1 (a_3 + sa_4)$$

Thus for every $n \geq n_0$ we have
$s(1 - \frac{a3}{s} - a_4) d_1 < (a_1 + a_2 + sa_4) d_1$
In the case when $a_1 + a_2 + a_3 + (s+1) a_4 < 1$ holds we obtain
$(1 - \frac{a3}{s} - a_4) \geq 1 - a_3 - a_4) > a_1 + a_2 + sa_4 \geq 0$
In the other case when $a_1 + a_2 + a_3 + (s+1) a_5 < 1$ holds, we obtain the same inequality by changing the sequence And hence $s\, d_1 < d_0$ for every $n \in N$ we can now use inequality and write
$F(s\, d_1) \leq F(d_0) - \tau$ for all n in N
Similarly we get
$\tau + F(s\, d(x_2, x_3)) \leq F(a_1 d(x_2, x_1)) + a_2 d(x_2, Sx_2) + a_3 d(x_1, T x_1) + a_4 d(x_2 Tx_1) + a_5 d(x_1, Sx_2)$
$\tau + F(s\, d(x_2, x_3)) \leq F(a_1 d(x_1, x_2)) + a_2 d(x_2, x_3) + a_3 d(x_1, x_2) + a_4 d(x_2, x_2) + a_5 d(x_1, x_3)$
$\tau + F(s\, d(x_2, x_3)) = F(a_1 d(x_1, x_2)) + a_2 d(x_2, x_3) + a_3 d(x_1, x_2) + a_5 \{d(x_1, x_2) + d(x_2, x_3)\}$
$\tau + F(s\, d(x_2, x_3)) \leq F(a_1 d(x_1, x_2)) + a_2 d(x_2, x_3) + a_3 d(x_1, x_2) + a_5 \{sd(x_1, x_2) + d(x_2, x_3)\}$
$\tau + F(s\, d(x_2, x_3)) \leq F(a_1 + a_3 + sa_5) d(x_1, x_2) + (a_2 + sa_5) d(x_2, x_3)$
$\tau + F(s\, d_2) \leq (a_1 + a_3 + sa_5) d_1 + (a_2 + sa_5) d_2$
since F is strictly increasing, and again by the same argument, this follows that
$s\, d_2 < (a_1 + a_3 + sa_5) d_1 + (a_2 + sa_5) d_2$
$s(1 - \frac{a_2}{s} - a_5) d_2 < (a_1 + a_3 + sa_5) d_1$
$s\, d_2 < d_1$
for every $n \in N$ we can now use inequality and write
$F(sd_2) \leq F(d_1) - \tau$
Similarly we can find
$F(sd_3) \leq F(d_2) - \tau$
Continuing in this way we will have
$F(sd_n) \leq F(d_{n-1}) - \tau$ *for all* $n \in N$
By condition F 4 we have
$F(s^n d_n) \leq F(s^{n-1} d_{n-1}) - \tau$ *for all* $n \in N$ *and hence by induction*

$$F(s^n d_n) \leq F(s^{n-1} d_{n-1}) - \tau \leq \ldots\ldots\ldots\ldots \leq F(d_0) - n\tau \qquad \ldots (3.2)$$

In the limit as $n \to \infty$, we get
$\lim_{n\to\infty} F(s^n d_n) = -\infty$ by F2
$\lim_{n\to\infty} (s^n d_n) = 0$
From condition (F3), there exists $k \in (0, 1)$ such that
$\lim_{n\to\infty} (s^n d_n)^k F(s^n d_n) = 0$
Multiplication of (3.2) with $(s^n d_n)^k$ yields
$0 \leq (s^n d_n)^k F(s^n d_n) + n(s^n d_n)^k \tau \leq (s^n d_n)^k F(d_0)$
Taking the limit as $n \to \infty$, we get
$\lim_{n\to\infty} n(s^n d_n)^k = 0$
Now, following the proof of Theorem 3.2 in [16], we can show that $\{x_n\}$ is a Cauchy sequence. Since (X, d, s) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$
Applying Lemma 2.3, we get
$\lim_{n\to\infty} d(x^*, x_n) = \lim\sup d(x^*, x_n) \leq sd(x^*, x^*) = 0.$
It remains to show that x^* is a common fixed point of S and T .first we show that x^* is a fixed point of S.,
Also, using (3.1), we have for all $n \in N$
$\tau + F(sd(Sx^*, x_{2n+2}) = \tau + F(sd(Sx^*, Tx_{2n+1}) \leq a_1 d(x^*, x_{2n+1}) + a_2\ d(x^*, Sx^*) + a_3 d(x_{2n+1}, Tx_{2n+1}) + a_4 d(x^*, Tx_{2n+1}) + a_5 (x_{2n+1}, Sx^*)$
$\leq F(a_1 (d(x^*, x_{2n+1}) + a_2 d(x^*, Sx^*) + a_3 d(x_{2n+1}, x_{2n+2}) + a_4 (x^*, x_{2n+2}) + a_5 (x_{2n+1}, Sx^*)$

Hence letting $n \to \infty$, we get (since $d(x^*, x_n) \to 0$)
$\tau + \lim_{n\to\infty} F(sd(Sx^*, x_{2n+2}) \leq -\infty$ This implies
$\lim_{n\to\infty} d(Sx^*, x_{2n+2}) = 0$ This implies
$d(Sx^*, x^*) = 0$
Thus we have, $Sx^* = x^*$.x^* is a fixed point of S. We also show that x^* is a fixed point of T. By proposition, we have

$\tau + F(sd(x_{2n+1}, Tx^*)) = \tau + F(sd(Sx_{2n}, Tx^*))$
$\leq F(a_1 d(x_{2n}, x^*)) + a_2 d(x_{2n}, Sx_{2n}) + a_3 d(x^*, Tx^*) + a_4 d(x_{2n}, Tx^*) \qquad + a_5 d(x^*, Sx_{2n})$
$\leq F(a_1 d(x_{2n}, x^*)) + a_2 d(x_{2n}, x_{2n+1}) + a_3 d(x^*, Tx^*) + a_4 d(x_{2n}, Tx^*) \qquad + a_5 d(x^*, x_{2n+1})$
Letting $n \to \infty$, we get
$\tau + \lim_{n\to\infty} F(sd(x_{2n+1}, Tx^*) \leq -\infty$.
This implies
$\lim_{n\to\infty} d(x_{2n+1}, Tx^*) = 0$.
$\Rightarrow d(x^*, Tx^*) = 0$.Thus we have , $Tx^* = x^*$. x^* is a fixed point of T. Therefore x^* is a common fixed point of S, T.
To show the uniqueness of the fixed point, suppose that, u be another common fixed point of S, T
$u = Su = Tu$ and $x^* = Tx^* = Sx^*$
with $x^* \neq u$. Then from (3.1)
$\tau + F(sd(Su, Tx^*) \leq F(a_1 d(u, x^*) + a_2 d(u, su) + a_3 d(x^*, Tx^*) \qquad + a_4 d(u, Tx^*) + a_5 d(x^*, Su))$
$\leq F(a_1 d(u, x^*) + a_4 d(u, Tx^*) + a_5 d(u, x^*))$
$\leq F((a_1 + a_4 + a_5) d(u, x^*))$
This implies that
$F(sd(u, x^*)) < F(d(u, x^*))$
which is a contradiction. Thus S and T have a unique common fixed point. This proves the result.

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