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Intuitionistic Fuzzy Hyponormal Operator in IFH-Space

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Abstract: In this work,we introduced the definition of Intuitionistic Fuzzy Hyponormal operator acting on an IFH-space, i.e.an operator $\mathbb{T} \in IFB(\mathcal{H})$ is Intuitionistic Fuzzy Hyponormalif $\|\mathbb{T}^* a\| \leq \|\mathbb{T}a\|$, $\forall a \in \mathcal{H}$or equivalently $\mathbb{T}^*\mathbb{T} - \mathbb{T}\mathbb{T}^* \geq 0$and given some elementary properties of Intuitionistic Fuzzy Hyponormal operator on an IFH-space. Also, we introduced some definitions like intuitionistic fuzzy invariant, eigenvalues, eigenvectors and eigenspaces which are related to Intuitionistic Fuzzy Hyponormal operator in IFH-space.
Keywords: Intuitionistic Fuzzy Adjoint operator (IFA-operator), Intuitionistic Fuzzy Hilbert space (IFH-space), Intuitionistic Fuzzy Hyponormal operator (IFHN-operator),Intuitionistic FuzzyInvariant (IF-invariant), Intuitionistic Fuzzy Normal operator (IFN-operator), IntuitionisticFuzzy Self-Adjoint operator (IFSA-operator).

I. INTRODUCTION

In 1986, Atanossov [11] introduced the notion of intuitionistic fuzzy set. Park [10] introduced the notion of intuitionistic fuzzy metric space ($\mathbb{T}$, M, N, *, ⋄) with the use of continuous t-norm * and continuous t-conorm ⋄ in 2004. Saadati and Park [17] introduced modulation of the intuitionistic fuzzy metric space in IFH-space using continuous t representable in 2005. The new idea of intuitionistic fuzzy normed spaces was introduced by Goudarzi et al. [13] and introduced the modified definition of intuitionistic fuzzy inner product space (IFIP-space) with the help of continuous t-representable ($\mathcal{T}$) in 2009. A triplet $(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ where $\mathcal{H}$is a real vector space, $\mathcal{T}$ is a continuous t -representable and $\mathcal{F}_{\mu,\nu}$ is an Intuitionistic Fuzzy set on $\mathcal{H}^2 \times \mathbb{R}$ which was introduced by Goudarzi et al. [13] in 2009,and also Majumdar and Samanta [15] gave the various definition of IFIP-space and some of their properties using$(\mathcal{H}, \mu, \mu^*)$.

The definition of IFH-space first introduced by Radharamani et al. [1] in 2018, and also some properties of IFA & IFSA operators in IFH-space by Radharamani et al.[2]. Then Radharamani et al. [3]introduced the concept of Intuitionistic Fuzzy Normal operator in 2020.An operator$\mathbb{T} \in IFB(\mathcal{H})$ if it commutes with its Intuitionistic fuzzy adjoint operator.i.e, $\mathbb{T}\mathbb{T}^* = \mathbb{T}^*\mathbb{T}$ and their properties. In 2020,Radharamani et al. [4],[5]giventhe definition of Intuitionistic Fuzzy Unitary operator (IFU-operator) and Intuitionistic Fuzzy Partial Isometry (IFPI-operator) on IFH-space $\mathcal{H}$, and gave some properties of these operators in IFH-space and also the relation with isometric isomorphism of $\mathcal{H}$ on to itself.

In this paper,we consider an Intuitionistic fuzzy normal operator in IFH-space and introduced the definition of Intuitionistic Fuzzy hyponormal operator(IFHN- operator) and we provided some important properties of IFHN- operator on IFH-space. And also introduce intuitionistic fuzzy invariant and eigenvectors and eigenspaces which is using in Intuitionistic Fuzzy Hyponormal Operator in IFH-space, which all are discussed in detail.

The classification of this paper is as follows:

Section 2 provides some preliminary definitions and theorems which are used in this paper.

In section 3, we introduced the concept of Intuitionistic Fuzzy hyponormal operator(IFHN- operator) and prove some properties of Intuitionistic fuzzy hyponormal operator have been studied.

II. PRELIMINARIES

Definition 2.1: [13] IFIP-space

Let $\mu: \mathcal{H}^2 \times (0, +\infty) \to [0,1]$ and $\nu: \mathcal{H}^2 \times (0, +\infty) \to [0,1]$ be Fuzzy sets, such that $\mu(u, v, t) + \nu(u, v, t) \leq 1, \forall\, u, v \in \mathcal{H}\ \&t > 0$. An Intuitionistic Fuzzy Inner Product Space (IFIP-Space) is a triplet $(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$, where $\mathcal{H}$ is a real vector space, $\mathcal{T}$ is a continuous t -representable and $\mathcal{F}_{\mu,\nu}$ is an Intuitionistic Fuzzy set on $\mathcal{H}^2 \times \mathbb{R}$ satisfying the following conditions for all u, $v, w \in \mathcal{H}$ and s, $r,\ t\ \in\ \mathbb{R}$:

(IFI – 1) $\mathcal{F}_{\mu,\nu}(u,\ v,\ 0) = 0$ and $\mathcal{F}_{\mu,\nu}(u,\ u,\ t) > 0$, for every $t > 0$.

(IFI - 2) $\mathcal{F}_{\mu,\nu}(u,\ v,\ t) = \mathcal{F}_{\mu,\nu}(v,\ u,\ t)$.

(IFI - 3) $\mathcal{F}_{\mu,\nu}(u,\ u,\ t) \neq \mathrm{H}(t)$ for some $t \in \mathbb{R}$ iff u$\neq$ 0,

$$\text{where } \mathrm{H}(t) = \begin{cases} 1, & \text{if } \ t > 0 \\ 0, & \text{if } \ t \leq 0 \end{cases}$$

(IFI - 4) For any $\alpha \in \mathbb{R}$,

$$\mathcal{F}_{\mu,v}(\alpha u, v, t) = \begin{cases} \mathcal{F}_{\mu,v}\left(u, v, \frac{t}{\alpha}\right), & \alpha > 0 \\ \mathrm{H}(t), & \alpha = 0 \\ \mathcal{N}_s\left(\mathcal{F}_{\mu,v}\left(u, v, \frac{t}{\alpha}\right)\right), & \alpha < 0 \end{cases}$$

(IFI - 5) $\sup\left\{\mathcal{T}\left(\mathcal{F}_{\mu,v}(u,\ w,\ s), \mathcal{F}_{\mu,v}(v,\ w,\ r)\right)\right\} = \mathcal{F}_{\mu,v}(u\ +\ v,\ w,\ t)$.

(IFI - 6) $\mathcal{F}_{\mu,v}(u,\ v,\ \cdot): \mathbb{R} \to [0,1]$ is Continuous on $\mathbb{R} \setminus \{0\}$.

(IFI - 7) $\lim_{t\to 0} \mathcal{F}_{\mu,v}(u,\ v,\ t)\ =\ 1$.

Definition 2.2: [1], [13] IFH-space

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFIP-Space with IP: $\langle u, v\rangle = \sup\left\{t \in \mathbb{R}: \mathcal{F}_{\mu,v}(u,v,t) < 1\right\}, \forall u, v \in \mathcal{H}$. If $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ is complete in the norm $\mathcal{P}_{\mu,v}$, then $\mathcal{H}$ is an Intuitionistic Fuzzy Hilbert Space (IFH-Space).

Definition 2.3: [2] IFA-operator

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-Space and let $\mathbb{P} \in$ IFB($\mathcal{H}$). Then there exists unique $\mathbb{P}^* \in$IFB($\mathcal{H}$) $\ni$ $\langle \mathbb{P}u, v\ \rangle\ =\ \langle\ u, \mathbb{P}^* v\ \rangle\ \forall\ u, v\ \in\ \mathcal{H}$.

Definition 2.4: [2] IFSA-operator

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-Space with IP:$\langle u, v\rangle = \sup\left\{\ t \in \mathbb{R}: \mathcal{F}_{\mu,v}(\ u,\ v,\ t) < 1\right\}, \forall\ u, v \in \mathcal{H}$ and let $\mathbb{P} \in$IFB($\mathcal{H}$). Then $\mathbb{P}$ is Intuitionistic Fuzzy Self-Adjoint Operator, if $\mathbb{P} =\ \mathbb{P}^*$, where $\mathbb{P}^*$ is Intuitionistic Fuzzy Self-Adjoint of $\mathbb{P}$.

Theorem 2.5: [2]

Let $(V, F_{\mu,\vartheta}, *)$ be an IFH – space with IP:$\langle x, y\rangle_\alpha^{N,M} = \sup\left\{u \in \mathbb{R}: F_{\mu,\vartheta}(x, y, u) < 1\right\} \forall\ x, y \in V$ and let S^* be the intuitionistic fuzzy adjoint operator of $S \in IFB(V)$. Then:

(i) $(S^*)^* = S$

(ii) $(\beta S)^* = \beta S^*$

(iii) $(\beta S + \gamma T)^* = \beta S^* + \gamma T^*$ where β, γ are scalars and $S \in IFB(V)$.

(iv) $(ST)^* = T^* S^*$.

Definition 2.6: [3] IFN-operator

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space with an IP: $\langle u, v\rangle = \sup\left\{t \in \mathbb{R}: \mathcal{F}_{\mu,v}(u,\ v,\ t) < 1\right\}, \forall\ u,\ v\ \in\ \mathcal{H}$ and let $\mathbb{P} \in$IFB($\mathcal{H}$). Then $\mathbb{P}$ is an Intuitionistic Fuzzy Normal Operator if it commutes with its IF-Adjoint. i.e.$\mathbb{P}\mathbb{P}^*\ =\ \mathbb{P}^*\mathbb{P}$.

Definition 2.7: [4] IFU-operator

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be a IFH-space with IP:$\langle u, v\rangle = \sup\left\{t \in \mathbb{R}: \mathcal{F}_{\mu,v}(u,v,t) < 1\right\} \forall u, v \in \mathcal{H}$ and let$\mathbb{P} \in IFB(\mathcal{H})$. Then $\mathbb{P}$ is an Intuitionistic fuzzy unitary operator if it satisfies$\mathbb{P}\mathbb{P}^* = I = \mathbb{P}^*\mathbb{P}$.

Definition 2.8: [4]Intuitionistic Fuzzy Isometric Isomorphism

Let *X* and *Y* be intuitionistic fuzzy normed linear spaces. An Intuitionistic Fuzzy isometric isomorphism of *X* into *Y* is a one to one linear transformation $\mathbb{P}$ of *X* into *Y* such that $\mathcal{P}_{\mu,v}(\mathbb{P}u, t) = \mathcal{P}_{\mu,v}(u, t)$ for every $u \in X$.

Theorem 2.9: [4]

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space with IP: $\langle u, v\rangle = \sup\left\{t \in \mathbb{R}: \mathcal{F}_{\mu,v}(u,v,t) < 1\ \right\} \forall u, v \in \mathcal{H}$ and let $\mathbb{P} \in IFB(\mathcal{H})$. If $\mathbb{P}$ is Intuitionistic Fuzzy Unitary operator if and only if it is an isometric isomorphism of $\mathcal{H}$ onto itself.

Definition 2.10: [13] IF-orthogonal

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space. $u, v \in \mathcal{H}$is saidto be IF-orthogonal to each other if $\mathcal{F}_{\mu,v}(u, v, t) =$ H(t), for each $t \in \mathbb{R}$ and it is denoted by $u \perp v$.

Theorem 2.11: [13]

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space. The orthogonality has the following properties:

(1) $0 \perp u, \forall\ u \in \mathcal{H}$.

(2) If $u \perp v$ then $v \perp u$.

(3) If $u \perp v$ then $u = 0$.

(4) If $u \perp u_i$ $(i = 1,2, \dots, n)$ then $u \perp (\sum_{i=1}^{n} u_i)$.

(5) If $u \perp v$ then for any $\alpha \in \mathbb{R}, u \perp \alpha v$.

(6) Let $\mathcal{F}_{\mu,v}$ be IF-continuous. If $u_n \xrightarrow{\tau_F} u, v \perp u_n$ $(n = 1,2, \dots)$ then $v \perp u$.

Definition 2.12: [13]

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space and $\mathcal{M} \subset \mathcal{H}$. $\mathcal{M}^{\perp}$ is the set of all $v \in \mathcal{H}$that are orthogonal to every $u \in \mathcal{M}$.

Theorem 2.13: [13]

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space, $\mathcal{F}_{\mu,v}$ be IF-continuous and $\mathcal{M}$ be a subset of $\mathcal{H}$. Then $\mathcal{M}^{\perp}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{M} \cap \mathcal{M}^{\perp} = \{0\}$.

Theorem 2.14: [13] The Pythagorean Theorem

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space and let $u \perp v$. Then $\mathcal{P}_{\mu,v}(u+v,t) = \mathcal{T}(\mathcal{P}_{\mu,v}(u,t), \mathcal{P}_{\mu,v}(v,t))$.

Definition 2.15: [5] Intuitionistic Fuzzy Projection operator

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space. $\mathcal{H}$can be decomposed into $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, i.e. for any $u \in \mathcal{H}$, $u = v \oplus w$ where $v \in \mathcal{M}$&$w \in \mathcal{M}^{\perp}$. An operator $\mathbb{P}$from $\mathcal{H}$ onto $\mathcal{M}$ is said to be IF-projection if $\mathbb{P}u = v$. It is denoted by $\mathbb{P}_{\mathcal{M}}$.

Note 2.16: [5]

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space and $\mathcal{M} \subset \mathcal{H}$be a closed subspace. The IF-orthogonal projection (IF-Projection operator) of $\mathcal{H}$ onto $\mathcal{M}$ is an operator from $\mathcal{H}$onto itself such that for $u \in \mathcal{H}$, $\mathbb{P}_{\mathcal{M}}u$ is the unique element in $\mathcal{M}$, i.e. $\mathbb{P}_{\mathcal{M}}u = v, v \in \mathcal{M}$.

Definition 2.17: [5] Intuitionistic Fuzzy Partial isometry operator

An operator $\mathbb{P} \in IFB(\mathcal{H})$ is said to be Intuitionistic Fuzzy (IF) partial isometry operator if there exists a closed subspace $\mathcal{M}$ such that $\mathcal{P}_{\mu,v}(\mathbb{P}u,t) = \mathcal{P}_{\mu,v}(u,t)$ for any $u \in \mathcal{M}$ and $\mathbb{P}u = 0$, for any $u \in \mathcal{M}^{\perp}$, here $\mathcal{M}$ is said to be the initial space of $\mathbb{P}$ and $\mathcal{N} = \mathcal{R}(\mathbb{P})$ is said to be the final space of $\mathbb{P}$.

III. MAIN RESULTS

In this section we introduced the definition of intuitionistic fuzzy hyponormal operator on IFH-space and some properties. Before that we introduced some preliminary definitions and theorems which are used to characterize intuitionistic fuzzy hyponormal operator.

Definition 3.1:

Let $(\mathcal{H}, \mathcal{F}_{\mu,v}, \mathcal{T})$ be an IFH-space and let$\mathbb{T} \in IFB(\mathcal{H})$. Then

(a) A scalar λ, $0 < \lambda < 1$, is called an eigenvalue of $\mathbb{T}$ if there exists non-zero $a \in \mathcal{H}$, such that $\mathbb{T}a = \lambda a$.

(b) A non-zero vector $a \in \mathcal{H}$ is called eigenvector of $\mathbb{T}$, if thereexists λ, $0 < \lambda < 1$, such that $\mathbb{T}a = \lambda a$.

Remark 3.2:

Corresponding to an eigenvalue λ there may correspond more than one eigenvector.

Theorem 3.3:

Let $\mathbb{T}$be an IFN-operator on a finite dimensional IFH-space $\mathcal{H}$ over $\mathbb{R}$, then

(i) $\mathbb{T} - \lambda\mathrm{I}$ is Intuitionistic fuzzy normal.

(ii) Every eigenvector of $\mathbb{T}$ is also an eigenvector of $\mathbb{T}^*$.

Proof:

(i) Since $\mathbb{T}$ is an IFN-operator, we have $\mathbb{T}\mathbb{T}^* = \mathbb{T}^*\mathbb{T}$

Also, $(\mathbb{T} - \lambda I)^* = \mathbb{T}^* - (\lambda I)^* = \mathbb{T}^* - \bar{\lambda} I$.

So, $(\mathbb{T} - \lambda I)(\mathbb{T} - \lambda I)^* = (\mathbb{T} - \lambda I)(\mathbb{T}^* - \bar{\lambda} I) = \mathbb{T}\mathbb{T}^* - \bar{\lambda}\mathbb{T} - \lambda\mathbb{T}^* - \lambda\bar{\lambda}$ … (3.1)

And $(\mathbb{T} - \lambda I)^*(\mathbb{T} - \lambda I) = (\mathbb{T}^* - \bar{\lambda} I)(\mathbb{T} - \lambda I) = \mathbb{T}^*\mathbb{T} - \lambda\mathbb{T}^* - \bar{\lambda}\mathbb{T} - \lambda\bar{\lambda}$ … (3.2)

Therefore, from (3.1) and (3.2) we get

$$(\mathbb{T} - \lambda I)^*(\mathbb{T} - \lambda I) = (\mathbb{T} - \lambda I)(\mathbb{T} - \lambda I)^*$$

Thus $\mathbb{T} - \lambda\mathrm{I}$ is an IFN-operator.

(ii) Let $a \in \mathcal{H}$ be an eigenvector of $\mathbb{T}$ corresponding to eigenvalue λ.

Which implies that, $\mathbb{T}a = \lambda a$.

Now,

$$\begin{aligned}\sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}(\mathbb{T}a, \mathbb{T}a, s) < 1\} &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}(a, \mathbb{T}^*\mathbb{T}a, s) < 1\} \\ &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}(a, \mathbb{T}\mathbb{T}^* a, s) < 1\} \quad \text{[since, T is Fuzzy Normal operator]} \\ &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}(\mathbb{T}^* a, \mathbb{T}^* a, s) < 1\}\end{aligned}$$

Since $\mathbb{T} - \lambda\mathrm{I}$ is an IFN-operator, therefore $a \in \mathcal{H}$, we have

$$\sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}((\mathbb{T} - \lambda I)a, (\mathbb{T} - \lambda I)a, s) < 1\} = \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}((\mathbb{T} - \lambda I)^* a, (\mathbb{T} - \lambda I)^* a, s) < 1\}$$

Since $\mathbb{T}a = \lambda a \implies \mathbb{T}a = \lambda I a \implies \mathbb{T}a - \lambda I a = 0 \implies (\mathbb{T} - \lambda I)a = 0$

Therefore, $\mathbb{T} - \lambda I = 0$.

Then $\sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}((\mathbb{T} - \lambda I)a, (\mathbb{T} - \lambda I)a, s) < 1\} = 0$

$$\implies \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,v}((\mathbb{T} - \lambda I)^* a, (\mathbb{T} - \lambda I)^* a, s) < 1\} = 0$$

Then,$(\mathbb{T} - \lambda I)^* = 0$.

Then for each $a \in \mathcal{H}$, we have $(\mathbb{T} - \lambda I)^* a = 0$

$$\implies \mathbb{T}^* a - \bar{\lambda} I a = 0 \implies \mathbb{T}^* a = \bar{\lambda} I a \implies \mathbb{T}^* a = \bar{\lambda} a$$

Therefore, a is eigenvector of $\mathbb{T}$ corresponding to eigenvalue $\bar{\lambda}$.

Definition 3.4: Intuitionistic Fuzzy Invariant (IF-Invariant)

Let $(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$be an IFNL-space and let $\mathbb{T} \in IFB(\mathcal{H})$. A subspace $\mathcal{M}$ of an IFNL-space $\mathcal{H}$ is said to be IF-invariant under $\mathbb{T}$, if $\mathbb{T}\mathcal{M} \subset \mathcal{M}$.

Theorem 3.5:

Let $\mathcal{M}$ be a closed subspace of an IFH-space and let $\mathbb{T} \in IFB(\mathcal{H})$. Then $\mathcal{M}$ is IF-invariant under $\mathbb{T}$ if and only if $\mathcal{M}^{\perp}$ is IF-invariant under $\mathbb{T}^*$.

Proof:

Suppose $\mathcal{M}$ is IF-invariant under $\mathbb{T}$.
Let $b \in \mathcal{M}^{\perp}$. We have to prove that$\mathbb{T}^* b \in \mathcal{M}^{\perp}$.
Let $a \in \mathcal{H}$. Since $\mathcal{M}$ is IF-invariant under $\mathbb{T} \implies \mathbb{T}a \in \mathcal{M}$.
Since $b \in \mathcal{M}^{\perp} \implies \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}a, b, s) < 1\} = 0$
$\implies \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a, \mathbb{T}^* b, s) < 1\} = 0$
Thus, $\mathbb{T}^* b \in \mathcal{M}^{\perp}$.
Conversely, suppose that $\mathcal{M}^{\perp}$ is IF-invariant under $\mathbb{T}^*$.
Since $\mathcal{M}^{\perp}$ is closed subspace of an IFH-space $\mathcal{H}$by theorem (2.13) and since $\mathcal{M}^{\perp}$ is IF-invariant under $\mathbb{T}^*$, therefore by above case $(\mathcal{M}^{\perp})^{\perp}$ is IF-invariant under $(\mathbb{T}^*)^*$.
But $(\mathcal{M}^{\perp})^{\perp} = \mathcal{M}$ and $(\mathbb{T}^*)^* = \mathbb{T}$.
Therefore, $\mathcal{M}$ is IF-invariant under $\mathbb{T}$.

Definition 3.6:

Let $\mathcal{M}$ be a closed subspace of an IFH-space and let $\mathbb{T} \in IFB(\mathcal{H})$. If both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are IF-invariant under $\mathbb{T}$, we say that $\mathcal{M}$ reduces $\mathbb{T}$ (or $\mathbb{T}$ is reduced by $\mathcal{M}$).

Theorem 3.7:

A closed subspace $\mathcal{M}$ of an IFH-space $\mathcal{H}$ reduces an operator $\mathbb{T}$ if and only if $\mathcal{M}$ is IF-invariant under both $\mathbb{T}$ and $\mathbb{T}^*$.

Proof:

Let us assume that $\mathcal{M}$ reduces an operator $\mathbb{T}$.
By the definition of reducibility, $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are IF-invariant under $\mathbb{T}$.
By theorem (3.5), if $\mathcal{M}^{\perp}$ is IF-invariant under $\mathbb{T}$, then $(\mathcal{M}^{\perp})^{\perp}$ i.e. $\mathcal{M}$ is IF-invariant under $\mathbb{T}^*$.
Thus, $\mathcal{M}$ is IF-invariant under both $\mathbb{T}$ and $\mathbb{T}^*$.
Conversely, suppose that $\mathcal{M}$ is IF-invariant under both $\mathbb{T}$ and $\mathbb{T}^*$.
Since $\mathcal{M}$ is IF-invariant under $\mathbb{T}^*$, $\mathcal{M}^{\perp}$ is IF-invariant under $(\mathbb{T}^*)^*$.
i.e. $\mathcal{M}^{\perp}$ is IF-invariant under $\mathbb{T}$.
Therefore, both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are IF-invariant under $\mathbb{T}$.
Thus, $\mathcal{M}$ reduces $\mathbb{T}$.

Definition 3.8:

Let $(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-space, $\mathbb{T} \in IFB(\mathcal{H})$ and let λ be an eigenvalue of $\mathbb{T}$. Then the set of all eigenvectors corresponding to λ together with 0 vector is called an eigenspace of $\mathbb{T}$ corresponding to the eigenvalue λ and is denoted by $\mathcal{M}_\lambda$.

Note3.9:

(1) By the definition, an eigenvector cannot be a zero vector. Therefore, $\mathcal{M}_\lambda$ necessarily contains some non-zero vectors.

(2) From (1), a non-zero vector $a \in \mathcal{M}_\lambda$iff$\mathbb{T}a = \lambda a$. Also $0 \in \mathcal{M}_\lambda$, the vector 0 definitely satisfies the equation $\mathbb{T}a = \lambda a$.
Therefore, $\mathcal{M}_\lambda = \{a \in \mathcal{H}: \mathbb{T}a = \lambda a\} = \{a \in \mathcal{H}: (\mathbb{T} - \lambda I)a = 0\}$.
Thus, $\mathcal{M}_\lambda$ is the Null-space of $\mathbb{T} - \lambda I$ on $\mathcal{H}$. Hence $\mathcal{M}_\lambda$ is a subspace of $\mathcal{H}$.

(3) Let $a \in \mathcal{H}$. Since $\mathcal{M}_\lambda$ is a subspace of $\mathcal{H}$ and λ is a scalar, then $\lambda a \in \mathcal{M}_\lambda$.
Since $a \in \mathcal{M}_\lambda \implies \mathbb{T}a = \lambda a \implies \mathbb{T}a \in \mathcal{M}_\lambda \implies \mathcal{M}_\lambda$ is IF-invariant under $\mathbb{T}$.

From (1), (2) and (3), $\mathcal{M}_\lambda$ is non-zero subspace of $\mathcal{H}$ invariant under $\mathbb{T}$.

Theorem 3.10:

If $\mathbb{T}$ be an IFN-operator on n-dimensional IFH-space $\mathcal{H}$, then each eigenspace reduces $\mathbb{T}$.

Proof:

Let $a_i \in \mathcal{M}_i$, the eigenspace of $\mathbb{T}$ and let λ_i be the corresponding eigenvalue. Then $\mathbb{T}a_i = \lambda_i a_i$.
Since $\mathbb{T}$ is an IFN-operator, then by theorem (3.3)$\overline{\lambda_i}$ is the eigenvalue for $\mathbb{T}^*$ (i.e. $\mathbb{T}^* a_i = \overline{\lambda_i} a_i$).
Since $\mathcal{M}_i$ is a subspace of $\mathcal{H} \implies \overline{\lambda_i} a_i \in \mathcal{M}_i \implies \mathbb{T}^* a_i \in \mathcal{M}_i$.
Therefore, $\mathcal{M}_i$ is IF-invariant under $\mathbb{T}^*$.
But $\mathcal{M}_i$ is IF-invariant under $\mathbb{T}$.

Thus, by theorem (3.5), $\mathcal{M}_i$ reduces $\mathbb{T}$.

Definition 3.11: Intuitionistic Fuzzy Hyponormal Operator (IFHN-operator)

Let $(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-space with IP: $\langle a, b\rangle = \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a, b, s) < 1\}$ $\forall\, a, b \in \mathcal{H}$ and let $\mathbb{T} \in IFB(\mathcal{H})$. Then $\mathbb{T}$is an intuitionistic fuzzy hyponormal (IFHN) operator on $\mathcal{H}$ if $\mathcal{P}_{\mu,\nu}(\mathbb{T}^* a, s) \leq \mathcal{P}_{\mu,\nu}(\mathbb{T} a, s), a \in \mathcal{H}$ or equivalently $\mathbb{T}^*\mathbb{T} - \mathbb{T}\mathbb{T}^* \geq 0$.

Theorem 3.12:

Let $(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-space with IP: $\langle a, b\rangle = \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a, b, s) < 1\}$ $\forall\, a, b \in \mathcal{H}$ and let $\mathbb{T} \in IFB(\mathcal{H})$ be an intuitionistic fuzzy hyponormal (IFHN) operator on $\mathcal{H}$. Then $\mathcal{P}_{\mu,\nu}((\mathbb{T} - zI)a, s) \geq \mathcal{P}_{\mu,\nu}\big((\mathbb{T}^* - \bar{z}I)a, s\big), a \in \mathcal{H}$, i.e. $\mathbb{T} - zI$ is an IFHN-operator.

Proof:

Given $\mathbb{T}$ is an IFHN-operator on $\mathcal{H}$.

Let $\mathcal{P}_{\mu,\nu}{}^2\big((\mathbb{T} - zI)a, s\big) = \langle(\mathbb{T} - zI)a, (\mathbb{T} - zI)a\rangle$

$$= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\big((\mathbb{T} - zI)a, (\mathbb{T} - zI)a, s\big) < 1\}$$

$$\geq \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a, (\mathbb{T} - zI)^*(\mathbb{T} - zI)^* a, s) < 1\} \text{ [since, by def. of IFHN-operator]}$$

$$= \langle(\mathbb{T} - zI)^* a, (\mathbb{T} - zI)^* a\rangle$$

$$\therefore \mathcal{P}_{\mu,\nu}{}^2\big((\mathbb{T} - zI)a, s\big) \geq \mathcal{P}_{\mu,\nu}{}^2((\mathbb{T} - zI)^* a, s)$$

$$\Rightarrow \mathcal{P}_{\mu,\nu}((\mathbb{T} - zI)a, s) \geq \mathcal{P}_{\mu,\nu}((\mathbb{T} - zI)^* a, s)$$

i.e. $\mathcal{P}_{\mu,\nu}((\mathbb{T} - zI)a, s) \geq \mathcal{P}_{\mu,\nu}(\overline{(\mathbb{T} - zI)}a, s)$

Thus, $\mathcal{P}_{\mu,\nu}((\mathbb{T} - zI)a, s) \geq \mathcal{P}_{\mu,\nu}\big((\mathbb{T}^* - \bar{z}I)a, s\big)$

Theorem 3.13:

Let$(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$be anIFH-space and let$\mathbb{T} \in IFB(\mathcal{H})$be an IFHN-operator on $\mathcal{H}$. Then $\mathbb{T}a = \lambda a \Rightarrow \mathbb{T}^* a = \bar{\lambda} a$.

Proof:

Let $a$be an eigenvector of $\mathbb{T}$ corresponding to the eigenvalue λ.

$$\Rightarrow \mathbb{T}a = \lambda a$$

Now,

$$\sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}a, \mathbb{T}a, s) < 1\} = \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a, \mathbb{T}^*\mathbb{T}a, s) < 1\}$$

$$= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a, \mathbb{T}\mathbb{T}^* a, s) < 1\} \text{ [since, T is anIFN-operator]}$$

$$= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}^* a, \mathbb{T}^* a, s) < 1\}$$

Since,$\mathbb{T} - zI$ is intuitionistic fuzzy hyponormal, $a \in \mathcal{H}$.

$$\sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\big((\mathbb{T} - \lambda I)a, (\mathbb{T} - \lambda I)a, s\big) < 1\} \geq \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}((\mathbb{T} - \lambda I)^* a, (\mathbb{T} - \lambda I)^* a, s) < 1\}$$

Since $\mathbb{T}a = \lambda a$, which implies that

$$\mathbb{T}a = \lambda I a \;\Rightarrow \mathbb{T}a - \lambda I a = 0 \;\Rightarrow (\mathbb{T} - \lambda I)a = 0$$

$\therefore \mathbb{T} - \lambda I = 0$.

Then $\sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}\big((\mathbb{T} - \lambda I)a, (\mathbb{T} - \lambda I)a, s\big) < 1\} = 0, \;\; \forall\, a \in \mathcal{H}$ … (3.1)

$\Rightarrow \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}((\mathbb{T} - \lambda I)^* a, (\mathbb{T} - \lambda I)^* a, s) < 1\} \leq 0, \;\; \forall\, a \in \mathcal{H}$

From (3.1), $(\mathbb{T} - \lambda I)^* a = 0$. Then for each $a \in \mathcal{H}$,

$(\mathbb{T} - \lambda I)^* a = 0 \;\Rightarrow\; \big(\mathbb{T}^* - \bar{\lambda} I\big)a = 0 \;\Rightarrow\; \mathbb{T}^* a - \bar{\lambda} I a = 0 \;\Rightarrow\; \mathbb{T}^* a = \bar{\lambda} a$

Therefore, a is an eigenvector of $\mathbb{T}^*$ corresponding to eigenvalue $\bar{\lambda}$.

Theorem (3.14):

$\mathbb{T} \in IFB(\mathcal{H})$ is an IFHN-operator iff$\mathcal{P}_{\mu,\nu}(\mathbb{T}^* a, s) \leq \mathcal{P}_{\mu,\nu}(\mathbb{T}a, s)$, for all $a \in \mathcal{H}$.

Proof:

Assume $\mathbb{T}$ is an IFHN-operator. Then by definition, $\mathbb{T}^*\mathbb{T} - \mathbb{T}\mathbb{T}^* \geq 0$.

which implies that $\mathbb{T}^*\mathbb{T} \geq \mathbb{T}\mathbb{T}^*$.

i.e. $\mathbb{T}\mathbb{T}^* \leq \mathbb{T}^*\mathbb{T}$

Let $\mathcal{P}_{\mu,\nu}(\mathbb{T}^* a, s) \leq \mathcal{P}_{\mu,\nu}(\mathbb{T}\, a, s)$

$\Leftrightarrow \mathcal{P}_{\mu,\nu}{}^2(\mathbb{T}^* a, s) \leq \mathcal{P}_{\mu,\nu}{}^2(\mathbb{T}\, a, s)$

$\Leftrightarrow \langle\mathbb{T}^* a, \mathbb{T}^* a\rangle \leq \langle\mathbb{T}a, \mathbb{T}a\rangle$

$\Leftrightarrow \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}^* a, \mathbb{T}^* a, s) < 1\} \leq \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}a, \mathbb{T}a, s) < 1\}$

$\Leftrightarrow \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}\mathbb{T}^* a, a, s) < 1\} \leq \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}^*\mathbb{T}a, a, s) < 1\}$

$\Leftrightarrow \langle\mathbb{T}\mathbb{T}^* a, a\rangle \leq \langle\mathbb{T}^*\mathbb{T}a, a\rangle$

$\Leftrightarrow \langle(\mathbb{T}\mathbb{T}^* - \mathbb{T}^*\mathbb{T})a, a\rangle \leq 0$

$\Leftrightarrow \mathbb{T}\mathbb{T}^* - \mathbb{T}^*\mathbb{T} \leq 0$

$\Leftrightarrow \mathbb{T}\mathbb{T}^* \leq \mathbb{T}^*\mathbb{T}$

Theorem (3.15):

Let $\mathbb{T} \in IFB(\mathcal{H})$ be a fuzzy hyponormal with $\mathbb{T}a_1 = \lambda_1 a_1$, $\mathbb{T}a_2 = \lambda_2 a_2$ and $\lambda_1 \neq \lambda_2$ then $\langle a_1, a_2 \rangle = 0$.

Proof:

Since $\mathbb{T}$ be an intuitionistic fuzzy hyponormal operator with $\mathbb{T}a_1 = \lambda_1 a_1$, $\mathbb{T}a_2 = \lambda_2 a_2$ and $\lambda_1 \neq \lambda_2$ then by theorem (3.3) $\mathbb{T}^* a_1 = \overline{\lambda_1} a_1$ and $\mathbb{T}^* a_2 = \overline{\lambda_2} a_2$.

Let $\lambda_1 \langle a_1, a_2 \rangle = \langle \lambda_1 a_1, a_2 \rangle$

$$\begin{aligned} &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\lambda_1 a_1, a_2, s) < 1\} \\ &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\mathbb{T}a_1, a_2, s) < 1\} \\ &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a_1, \mathbb{T}^* a_2, s) < 1\} \\ &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a_1, \overline{\lambda_2} a_2, s) < 1\} \\ &= \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(\lambda_2 a_1, a_2, s) < 1\} \\ &= \langle \lambda_2 a_1, a_2 \rangle \\ &= \lambda_2 \langle a_1, a_2 \rangle \end{aligned}$$

Hence, if $\lambda_1 \neq \lambda_2$ then $\langle a_1, a_2 \rangle = 0$. i.e. $a_1 \perp a_2$.

Theorem (3.16):

Let $(\mathcal{H}, \mathcal{F}_{\mu,\nu}, \mathcal{T})$ be an IFH-space with IP: $\langle a, b \rangle = \sup\{s \in \mathbb{R}: \mathcal{F}_{\mu,\nu}(a, b, s) < 1\}\ \forall\, a, b \in \mathcal{H}$ and let $\mathbb{T} \in IFB(\mathcal{H})$ be an IFHN-operator on $\mathcal{H}$ with $\mathcal{M} \subset \mathcal{H}$IF-invariant under $\mathbb{T}$ also let $\mathbb{T}_{\mathcal{M}}$ be intuitionistic fuzzy hyponormal. Then $\mathcal{M}$ reduces $\mathbb{T}$.

Proof:

Let $a \in \mathcal{M}$, the eigenspace of $\mathbb{T}$and let the corresponding eigenvalue of $\mathbb{T}$ be λ.

So that $\mathbb{T}a = \lambda a$. Since $\mathbb{T}$ is an IFN-operator then by theorem (3.3), $\mathbb{T}^* a = \bar{\lambda} a,\ a \in \mathcal{H}$.

Since $\mathcal{M}$ is a subspace, $\bar{\lambda} a \in \mathcal{M} \Rightarrow \mathbb{T}^* a \in \mathcal{M}$.

$\Rightarrow \mathcal{M}$ is IF-invariant under $\mathbb{T}^*$, but $\mathcal{M}$ is IF-invariant under $\mathbb{T}$.

Hence, by theorem(3.7), $\mathcal{M}$ reduces $\mathbb{T}$.

Corollary 3.17:

Let $\mathbb{T}$ be an IFHN-operator on $\mathcal{H}$ and $\mathcal{M} = \{a \in \mathcal{H}: \mathbb{T}a = \lambda a\}$ then $\mathcal{M}$ reduces $\mathbb{T}$ and $\mathbb{T}_{\mathcal{M}}$ is intuitionistic fuzzy hyponormal.

Corollary 3.18:

Let $\mathbb{T}$ be an IFHN-operator on $\mathcal{H}$ and let $\mathcal{M} \subset \mathcal{H}$, IF-invariant under $\mathbb{T}$. Then $\mathbb{T}_{\mathcal{M}}$ is intuitionistic fuzzy hyponormal.

IV. CONCLUSION

Intuitionistic Fuzzy Hyponormal operator (IFHN- operator) on IFH-space is introduced which is new idea. And also discuss classic form of theorems play the role a prototype in our discussion of this paper. These relations are very new and helpful for the further study of functional analysis on intuitionistic fuzzy concept.Some properties of IFHN- operator have been investigated which is useful for the further research in applications of functional analysis in fuzzy and intuitionistic fuzzy concept.

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