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Analyzing the Brachistochrone Problem for the Conic Sections

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Abstract
This paper explores the brachistochrone problem for several curves. Throughout the paper, we find the time taken by an object of any mass to travel from the initial position (1, 0) to the final position (0, 1) through any curve. Finding a generalized integral time formula applicable to all curves of the form y = f (x), we further apply the formula to a line and the following conic sections - parabola of form $y = ax^2 + bx + c$ and $x = ay^2 + by + c$, , circle of the form $(x - a)^2 + (y - b) = r^2$, ellipse with x and y as major axes, and hyperbola with x and y as major axis. The parabola with equation $x = ay^2 + by + c$ was found to be the curve of fastest descent among all the conic sections.

I. Introduction

The brachistochrone problem, proposed by Johann Bernoulli in 1696, aims to find, for points A and B with A located above B, a curve such that descent of an object due to gravity along that curve takes the shortest possible time. The problem considers only gravitational force on the object and ignores other forces such as friction. It was solved by a number of famous mathematicians such as Johann Bernoulli himself, Jakob Bernoulli, and Isaac Newton, all of them proposing a different approach [4]. Till date, numerous researchers have tried to expand upon the problem by including external factors such as non-uniform gravitational field [1], friction [2] and some other challenges, examining the problem further. These methods have been quite effective but require heavy machinery such as the calculus of variations and the Euler-Lagrange differential equation. Additionally, all of the papers are predominantly focused on deriving the equation of the curve with quickest descent and are not focused on other curves. In this paper, we focus on a more elementary approach and study descent along the conic sections from a point A with coordinates $(0, 1)$ to a point B with coordinates $(1, 0)$. In Section 2, we utilise a general approach for calculating the time it takes to descend along a curve defined by a function $y = f(x)$. In the following section,3 we give a detailed account of the time period in cases of parabola, circle, ellipse, and hyperbola respectively. Using an online tool [5] to find the value of integrals, we compare the findings of time periods of different curves. The curves when arranged in decreasing order of their time periods are as follows : line > parabola of form $y = f(x)$ = hyperbola with y axis as major axis > ellipse with y axis as major axis > circle > ellipse with x axis as major axis = hyperbola with x axis as major axis > parabola of form $x = f(y)$.
The methodology mainly comprises of independent calculations for calculating the time. Additionally, a small literature review was done to gather historical background on the problem.

2 Generalized integral time formula for any curve and descent for a straight line

Let us take an object of mass m sliding from point A to point B. Throughout the paper, we will consider the initial position position(A) to be $(0, 1)$ and the final position(B) to be $(1, 0)$. In order to find the time taken, we will first use the law of conservation of energy. The energy at the initial point is just the gravitational potential energy mgh where h is the height. We have $h = 1$ since our initial point is (1,0). The energy at any other point P on the curve will be $\frac{1}{2}mv^2 + mgy$ where v is the velocity of the object at P and h is the y-coordinate. By conservation of energy,

$$mg * 1 = \frac{1}{2}mv^2 + mgy$$

$$\frac{1}{2}mv^2 = mg(1 - y)$$

$$v = \sqrt{2g(1 - y)}$$

We have found the velocity at any point. By definition, velocity is the derivative of displacement with respect to time i.e $v = \frac{ds}{dt}$. Rearranging gives $dt = \frac{ds}{v}$ and integrating we obtain

$$t = \int_0^1 dt = \int_0^1 \frac{ds}{v} = \int_0^1 \frac{ds}{\sqrt{2g(1 - y)}}$$

Note that ds is the small change in displacement in a small period of time dt. Our aim is to express ds in terms of dx . In the below graph, the blue line represents the curve $y = x^2$ and the purple line refers to the tangent to the curve at $x = 1$. The angle made by the tangent line with the positive x axis is denoted by θ . Hence, the slope of the tangent line is equal to $\tan\theta$. Through the right angled triangle formed with ds as the hypotenuse and dx as the base, $ds = \dfrac{dx}{\cos\theta}$.

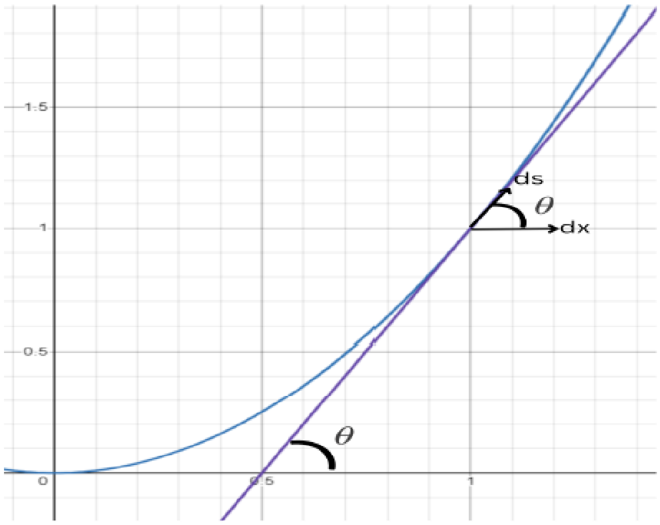

Figure 2.1: Graph of $y = x^2$ along with its tangent $y = 2x - 1$ at $x = 1$

Now, the derivative of the curve at $x = 1$ is the slope of the line i.e. $\tan\theta$. Now from equation $sec^2\theta = tan^2\theta + 1$ we have

$$cos^2\theta = \frac{1}{tan^2\theta + 1}.$$

As $\theta \leq \frac{\pi}{2}$, $\cos\theta$ is positive. Now

$$\cos\theta = \frac{1}{\sqrt{tan^2\theta + 1}} = \frac{1}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}}$$

So, $ds = \sqrt{\left(\left(\frac{dy}{dx}\right)^2 + 1\right)}dx$ and

$$t = \int_0^1 \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{2g(1-y)}}dx \tag{2.1}$$

This is the general time formula, which is applicable to any curve $y = f(x)$.

Consider the case of a line passing through the points (1,0) and (0,1). It is given by $y = 1 - x$ so that $\frac{dy}{dx} = -1$. Hence, the time taken by an object to roll down the line $y = x - 1$ from (1,0) to (0,1) is equal to

$$\int_0^1 \sqrt{\frac{1 + (-1)^2}{2g(1 - 1 + x)}}dx = \int_0^1 \sqrt{\frac{2}{2gx}}dx = \frac{2\sqrt{x}}{\sqrt{g}}\bigg|_0^1 = \frac{2}{\sqrt{g}} = 0.638877 \text{ seconds}$$

(Taking the value of g as 9.8).

3 Time taken for the descent by various conic sections

In this section, we discuss the time taken by an object of any mass to travel from $(1,0)$ to $(0,1)$ through various conic sections listed below.

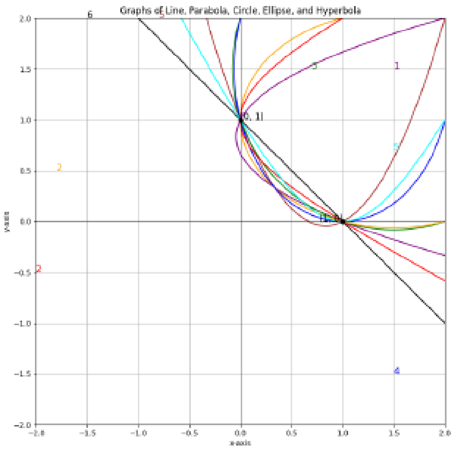

Figure 3.1: A graph summarising the curves, which are labelled in the order of increasing descent time.

The code for the above graph is accessible at https://github.com/Aarush2012/Brachistochrone-curve-graph.git.

3.1 Parabola

Consider a parabola of the form y= ax^2+bx+c passing through the points $(1,0)$ and $(0,1)$. Let us substitute the coordinates and find the equation of such a parabola. We get:

$$\begin{cases} a(1)^2 + b(1) + 1 = 0 \\ c = 1 \end{cases}$$

The general equation for a parabola passing these points is

$$y = ax^2 - (a+1)x + 1.$$

We can now apply the general formula (2.1). First, we differentiate: $\frac{dy}{dx} = 2ax - a - 1$. Substituting, we get

$$t(a) = \int_0^1 \sqrt{\frac{1+(2ax-a-1)^2}{2g(1-ax^2+(a-1)x-1)}}dx$$

Note that the time t is dependent on the value of the parameter a, and we can try to find the minimum of $t(a)$ to minimize the descent time.

We calculate the value of the above integral by plotting it into Desmos [5]. As a is a variable, we add a slider to change the value of a. The observations are that the value of the integral is the least when $a = 1$ with time taken for descent equal to 0.595227... seconds. At a=0.9, the value comes out be 0.5958 and at 1.1, it is 0.59543. From a=0 to a=1, the time taken decreases and after 1 ,it keeps on increasing. The curve is represented by $y = x^2 - 2x + 1 = (x-1)^2$

Now, let us discuss the time taken by a parabola of the form $x = ay^2 + by + c$. Substituting the coordinates in the equation, $c = 1$ and $a+b+1 = 0$ so that $b = -1-a$. The parabola is of the form $x = ay^2-(a+1)y+1$. Differentiating the equation with respect to y, $\dfrac{dx}{dy} = 2ay - a - 1$. Our second step is to express y in terms of x. For this, we will express x in terms of a perfectly squared equation, which will help us isolate y. We have

$$x = ay^2 - (a+1)y + \frac{(a+1)^2}{4a} + 1 - \frac{(a+1)^2}{4a} = \left(\sqrt{a}y - \frac{a+1}{2\sqrt{a}}\right)^2 + \frac{4a-(a+1)^2}{4a}$$

. Therefore

$$\pm\sqrt{\frac{4ax-4a+(a+1)^2}{4a}} = \sqrt{a}y - \frac{a+1}{2\sqrt{a}}$$

so that

$$y = \frac{\pm\sqrt{\frac{4ax-4a+(a+1)^2}{4a}} + \frac{a+1}{2\sqrt{a}}}{\sqrt{a}}$$

Clearly, the graph of the descent curve cannot pass through the second quadrant. In addition, it has to pass through the points $(0,1)$ and $(1,0)$. This implies that in the equation above we need to choose the minus sign and restict a to the interval $a \in [0,1]$. So we have

$$y = \frac{-\sqrt{\dfrac{4ax-4a+(a+1)^2}{4a}} + \dfrac{a+1}{2\sqrt{a}}}{\sqrt{a}} = \frac{a+1-\sqrt{4ax-4a+(a+1)^2}}{2a}$$

Now, we substitute the values of $\frac{dy}{dx}$ and y in the equation (2.1) , we obtain an integral depending on a. The least value of this integral is 0.58377.... which occurs at a=0.91.

3.2 Circle

Consider a circle of $(x-a)^2+(y-b)^2=r^2$ passing through the points (1,0) and (0,1). Here, a stands for the x-coordinate of the centre and b refers to the y coordinate of the centre; r is the radius of the circle. Substituting the points in the equation we get

$$\begin{cases}(1-a)^2+b^2=r^2\\ a^2+(1-b)^2=r^2\end{cases}$$

Equating the left sides, we have

$$1-2a+a^2+b^2=a^2+1-2b+b^2$$

so that $a=b$. Now the first equation gives $r^2=2a^2-2a+1$ and we have

$$(x-a)^2+(y-a)^2=2a^2-2a+1$$

We can rewrite it as follows:

$$x^2+y^2-2a(x+y-1)-1=0$$

Differentiating with respect to x, we get

$$2x+2y\frac{dy}{dx}-2a(1+\frac{dy}{dx})=0$$

so that

$$\frac{dy}{dx}=\frac{a-x}{y-a}$$

In the integral, all the y terms should be represented in terms of x so that we can integrate with respect to x. We express y as

$$y=-\sqrt{2a^2-2a+1-(x-a)^2}+a$$

We use the negative sign because it represents the lower part of the circle, which is used to traverse from (1,0) to (0,1). As for the parabola case, we substitute $\frac{dy}{dx}$ and y in equation (2.1). The least value of the integral is 0.58512.... which appears when $a=1.3$. An important point to consider here is that $a \geq 1$. This is due to the fact that when $a<1$ the circle no longer uses its lower part to pass through those points. Instead, the graph will be represented by $\{y=\sqrt{2a^2-2a+1-(x-a)^2}+a$,which is the upper part of the circle. Figure 3.2 graphically demonstrates the above idea.

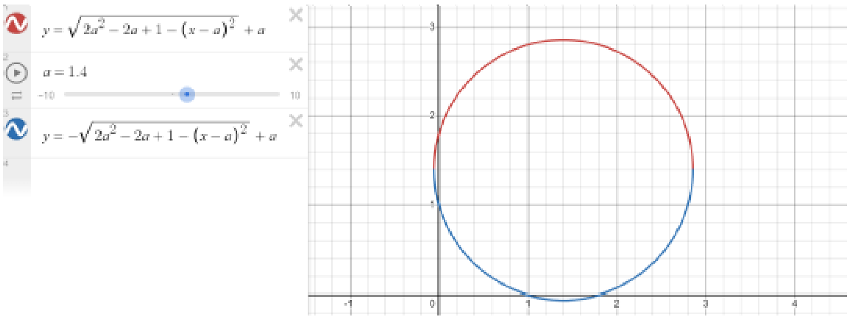

Figure 3.2: Curves represented graphically demonstrating why the negative sign was chosen for positive values of a

Now, for the case $a < 1$, we find the value of the integral by using the corresponding y value 3.2. The integral's value keeps on decreasing as x becomes more negative until it reaches a limiting value of 0.6388... seconds. This is same as the least time taken by a line as when $x \to -\infty$, the part of the circle joining (1,0) and (0,1) becomes increasingly flatter, eventually taking the shape of a line.

3.3 Ellipse

Consider an ellipse with x-axis as its major axis and x, y coordinate of its center as x_0 ,1. i.e. $\frac{(x-x_0)^2}{a^2} + \frac{(y-1)^2}{b^2} = 1$ passing through the points (1,0) and (0,1). Substituting the coordinates in the equation, we get

$$\begin{cases} \frac{(1-x_0)^2}{a^2} + \frac{1}{b^2} = 1 \\ \frac{x_0^2}{a^2} + \frac{0}{b^2} = 1 \to x_0^2 = a^2 \end{cases}$$

. Inserting the value of x_0 as $+a$ in the first equation, $\frac{(1-a)^2}{a^2} + \frac{1}{b^2} = 1$ So, $\frac{1}{b^2} = 1 - \frac{(1-a)^2}{a^2} = \frac{-1+2a}{a^2}$ $\therefore b^2 = \frac{a^2}{2a-1}$. So , the equation of the ellipse in this case is

$$(x-a)^2 + (y-1)^2(2a-1) = a^2 \tag{3.1}$$

We take the value of x as $+a$ as the ellipse is defined only for positive a values. . Differentiating the above equation with respect to x, $2(x-a) + 2(2a-1)(y-1)\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{a-x}{(2a-1)(y-1)}$$

. The next step will be to isolate y. $(y-1)^2 = \frac{a^2-(x-a)^2}{2a-1}$.

$$y = -\sqrt{\frac{2ax-x^2}{2a-1}} + 1. \tag{3.2}$$

. Due to the same reason in the curves discussed, we choose the negative sign. Plugging the respective values of $\frac{dy}{dx}$ and y into the time period equation 2.1, we find that value of this integral varies inconsistently for values of a. However, gradually, with a huge increase in the value of a, there is a decreasing trend in the value of this integral. Hence, as $a \to \infty$, the value of the integral reaches a limiting value. Let's go back to the original equation we derived 3.1. Expanding it, we obtain $x^2 - 2ax + a^2 + (2a-1)(y-1)^2 = a^2$. As our aim is to examine the curve near $x = 0$ and $x = 1$, x^2 can be neglected. $\therefore 2ax = (2a-1)(y-1)^2$. Now, $2a = 2a-1$ as we are finding the equation in which $a \to \infty$.

$$x = (y-1)^2 \tag{3.3}$$

We expressed an ellipse of type 3.1 with a significantly large value of a in the form of a parabola. Expressing the equation 3.3 in terms of x,

$$y = \pm\sqrt{x} + 1 \rightarrow y = -\sqrt{x} + 1 \tag{3.4}$$

. $\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$. Hence, the least time taken by an object to reach from $(1,0)$ to $(0,1)$ through 3.3= $\int_0^1 \sqrt{\frac{1+\frac{1}{4x}}{2*9.8(1+\sqrt{x}-1)}}dx = 0.58436....$ seconds.

Let us take the case of an ellipse with y axis as its major axis and center as (1,y_0). The equation of such an ellipse would be $\frac{(y-y_0)^2}{a^2} + \frac{(x-1)^2}{b^2} = 1$. Applying the former strategy,

$$\begin{cases} \frac{y_0^2}{a^2} = 1 \rightarrow y_0^2 = a^2 \\ \frac{(1-a)^2}{a^2} + \frac{1}{b^2} = 1 \rightarrow b^2 = \frac{a^2}{2a-1} \end{cases}$$

. Plugging these values in the equation, $\frac{(y-a)^2}{a^2} + \frac{(x-1)^2(2a-1)}{a^2} = 1$ Hence, $\{(y-a)^2 + (2a-1)(x-1)^2 = a^2$.
Differentiating with respect to x, $2(y-a)\frac{dy}{dx} + 2(2a-1)(x-1) = 0$

$$\frac{dy}{dx} = \frac{(1-2a)(x-1)}{(y-a)}$$

For all , $a \geq 1$

$$,y = -\sqrt{a^2 - (x-1)^2(2a-1)} + a \tag{3.5}$$

. Using the equation 2.1T to find the time formula, we find that the least time is observed to occur at a=1.375 which is 0.588349.... seconds. For $a > 2.2$, the value of the integral is seen to have values slightly more than 0.59, and for $a \in [1, 2.2]$, the value decreases until $a = 1.375$ and then starts to increase.

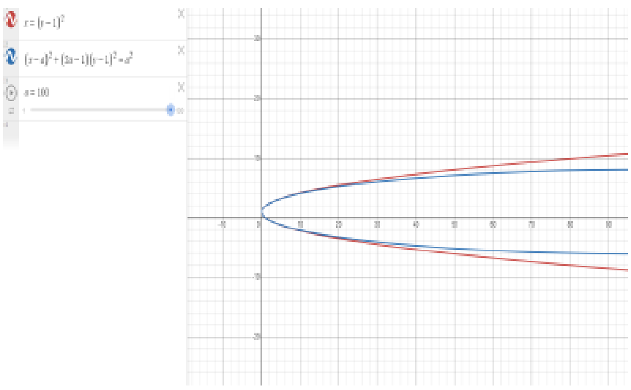

Figure 3.3: Graph of ellipse of type 3.1 with a=100 and parabola $x = (y-1)^2$

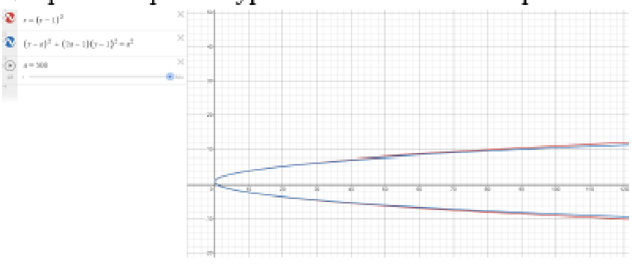

Figure 3.4: Graph of ellipse of type (3.1) with a=500 and parabola $x = (y-1)^2$

3.4 Hyperbola

In the previous section, while deriving the equation of the ellipse 3.1, we equated x to $+a$ because when a becomes negative, the ellipse takes the form of a hyperbola. Hence, for a hyperbola with x axis as its major axis and center as $(x_0, 1)$, the time taken by an object to pass through it can be defined as 3.3 with negative values of a.

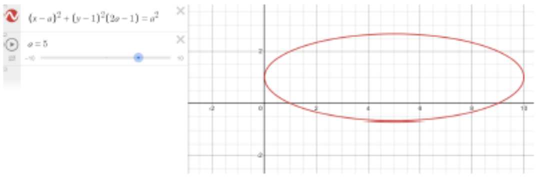

Figure 3.5: Graph of equation 3.1 with positive value of a

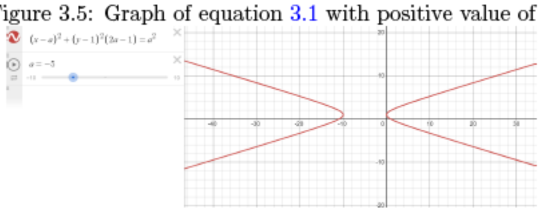

Figure 3.6: Graph of equation 3.1 with negative value of a

This integral also has a gradually decreasing trend with a large decrease in the value of a. Hence ,as $a \to -\infty$, the integral approaches a limiting value. In the equation 3.2, x^2 can be neglected as it is drastically smaller than $2ax$. Also, $2a$ can be equated to $2a-1$ as $a \to -\infty$. Hence, the equation becomes

$$y = 1 - \sqrt{x}$$

This is same as the equation we derived for the ellipse 3.4. Hence, the least time will be same as that of the ellipse with x axis as major axis i.e. 0.58436 seconds.

Now, for an object to pass through a hyperbola with y axis as major axis and centre at $(1, y_0)$ from $(1,0)$ to $(0,1)$, we will again choose the negative values of a in the equation for the ellipse 3.3 with y axis as major axis. However, the same curve used for the ellipse won't be applicable to the hyperbola. This is due to the fact that the curve 3.5 which passes through $(1,0)$ and $(0,1)$ does not through pass through those points for negative values of a.

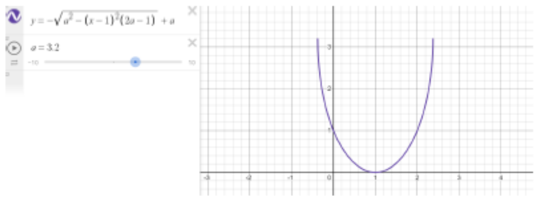

Figure 3.7: Graph of equation3.5 for positive values of a

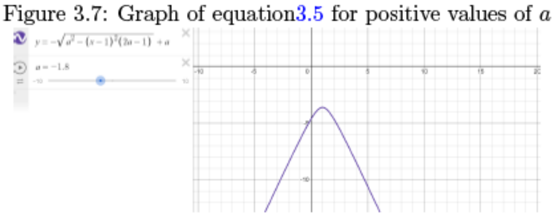

Figure 3.8: Graph of equation 3.5
for negative values of a

Instead, the upper part i.e.

$$y = \sqrt{a^2 - (x-1)^2(2a-1)} + a \tag{3.6}$$

will be the equation for the hyperbola. Finding the value of the time integral 2.1 by plugging the values of $\dfrac{dy}{dx}$ and y , we find that for all negative values of a, we find that there is also a decreasing trend in the its value. As, a approaches $-\infty$, the integral reaches a limiting value. Rearranging equation 3.6, $(y-a)^2 = a^2 - (x-1)^2(2a-1) \rightarrow y^2 - 2ay + a^2 = a^2 - (x-1)^2(2a-1)$. We want to work with the curve near the points $(0,1)$ and $(1,0)$.Hence, y^2 can be considered negligible. $\therefore 2ay = (x-1)^2(2a-1)$. $2a = 2a-1$ as $a \rightarrow -\infty$. Hence, the curve will be $y = (x-1)^2$. $\frac{dy}{dx} = 2(x-1)$. So, the time integral =

$$\int_0^1 \sqrt{\frac{1+4(x-1)^2}{2*9.8(1-(x-1)^2)}}dx$$

The least time taken by an object of any mass to travel from the point $(1,0)$ to $(0,1)$through hyperbola 3.6, $= 0.595227$ seconds.

4 Conclusion

Comparing the time taken by an object to pass through a line and various conic sections, we observe that a parabola of the form $x = f(y)$ is the curve with the quickest descent. However, we have restricted the number of parameters in the cases of ellipse and hyperbola by assuming them to be centered at $(x_0, 1)$ and $(1, y_0)$ for those with x and y axes as major axis respectively. Further studies could be done to validate this paper's findings and expand upon the research by finding the least time to traverse any ellipse and hyperbola. Additionally, using the parametric equations of a general cycloid, a similar method can be employed to prove why the cycloid is the curve of quickest descent.

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