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Complex-Order Fractional Modelling of a System

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Abstract
The aim of our research is to transform a non-linear model into a complex-order fractional model adapted from the third-generation of CRONE controller. We used the differential equation governing the angles of an X8-configuration drone to demonstrate this transformation. We used the Taylor series expansion and the fractional-order Taylor series expansion of two differential equations to approximate the exact model. The effectiveness of this model was demonstrated through simulation in MATLAB.

I. Introduction

Fractional calculus is regarded as a long-standing field of study that originated in the 17 th century. A hundred years later, fractional calculus attracted the attention of renowned mathematicians such as P.S. Laplace (1812), J.B.J. Fourier (1882), N.H. Abel (1803-1826), J. Liouville (1832-1873), B.Riemann(1847), H. Holmgren (1865-1867), A.K. Grunwald (1867-1872) [1]. It remained an unpopular subject amongst the engineering community, but its significance was only recognized in the second half of the 20th century following the development of the computer tools that made it possible to envisage applications in the field of engineering sciences. Researchers first applied it to the regulation and control of the systems. Fractional order systems are used as reference systems for control. There are three main reasons for employing non-integer controllers or using control methodologies based on non-integer differentiation **[1]**.

Furthermore, natural phenomena exhibiting dynamic behavior of this type are common . Such phenomena are also encountered in the analysis of propagation and vibration phenomena (even in their most idealized from)**[1]**.

Finally, the specific dynamic properties of fractional differential equations enable the development of highly interesting control strategies, even in the case of ordinary models, particularly in terms of robustness against model uncertainties, or in field of « adaptive impedance » stabilization through absorption **[1]**.

A number of concrete applications, including those in an industrial context have definitely demonstrated the practical value of these mathematical objects, which are still relatively uncommon but are relevant to virtually all disciplines.

However, like any far-reaching tool that has not yet reached a sufficient stage of maturity and unification, fractional differential operators exhibit great variety in terms of their mathematical representation, analysis and approximation techniques, numerical simulation, etc…

One of the applications of fractional modeling is CRONE control. CRONE control can be used to control non-linear, chaotic and highly unstable systems, and the most robust method is third-generation CRONE. This type of control use fractional operators of complex order; therefore, our model must be a fractional model of complex order to be suitable for this application [2]. This is why are we will transform the non-linear models into fractional-order linear models whilst preserving the initial conditions and the shape of the curve.

II. Méthode

- **Outils**

Fractional calculus is the field of mathematical analysis concerned with the investigation and application of integrals and derivatives of arbitrary order. Fractional calculus can be regarded as both a long-established and a relatively new subject. In recent years, considerable interest in fractional calculus has been stimulated by its applications in various fields of physics and engineering.[3][4]

Fractional calculus is generalization of integration and differentiation to the fundamental operator of non-integer order ${}_{t0}D_t^m$, where $\mathbf{t_0}$ and $\mathbf{t_1}$ are the limits of the operation.
The continuous integro-differential operator is defined by :

$$ {}_{t0}D_t^m = \begin{cases} \frac{d^m}{dt^m} & pour\ R(m) > 0 \\ 1 & pour\ R(m) = 0 \\ \int_{t_0}^{t} (d\tau)^{-m} & pour\ R(m) < 0 \end{cases} \quad (1) $$

Where m ∈ C is the order of the operation, and C is the ring of complex numbers.
The symbol R (.) denotes the real part of complex number.[5]
The three most commonly used definitions for the generalization of derivative and the fractional integral are the Grunwald Leitnikov definition, Riemann-Liouville definition, and the Caputo definition.[5]

- **Steps to follow**

-solve the model's differential equation
-Solve the fractional differential equation
-compare the results using the solution frameword

- **Model of $\dot{x}_2 = -1.4355 \times 10^{-1} x_2^2 + 65.51 U_\Phi$**

The solution to this differential equation in terms of U_Φ for the initial condition x_2=0 is

$$ x_2 = \sqrt{456.36 U_\Phi} \left(\frac{1 - e^{-6.1332\sqrt{U_\Phi} t}}{1 + e^{-6.1332\sqrt{U_\Phi} t}} \right) \quad (2) $$

Let us use the Taylor series expansion of x_2 to obtain

$$ x_2(t) = 262 U_\Phi t + 803.5 U_\Phi^{\frac{3}{2}} t^2 + 1643 {U_\Phi}^2 t^3 + \ldots \quad (3) $$

The formula for the complex-order fractional integrator with a right-hand side is given by

$$ D^{Re+iIm} x = ax + 2\,a \times \sqrt{456.36 U_\Phi} \quad (4) $$

Using the fractional Taylor series expansion of the solution to this differential equation , we obtain[6][7]

$$ X = 2 \times \sqrt{456.36 U_\Phi} \left(\frac{\sqrt[\alpha]{at}}{1!} + \frac{\left(\sqrt[\alpha]{at}\right)^2}{2!} + \frac{\left(\sqrt[\alpha]{at}\right)^3}{3!} + \ldots \right) \quad (5) $$

Simplifying $v a^{\frac{-i\beta}{\alpha^2+\beta^2}}$, we obtain

$$ a^{\frac{-i\beta}{\alpha^2+\beta^2}} = \exp\left(\ln\left(a^{\frac{-i\beta}{\alpha^2+\beta^2}}\right)\right) \quad (6) $$

Where α is the real part of the fractional model and β is the imaginary part of the fractional model
By comparing the two Taylor series expansion and using the previous equality, we obtain

$$ \left(e^{\frac{\alpha}{\alpha^2+\beta^2}}\right) \cos\left(\frac{\beta}{\alpha^2+\beta^2}\right) = 6.1332\sqrt{U_\Phi} \quad (7) $$

According to the relationship, Re and Im depend on the input $\sqrt{U_\theta}$ but let us choose input values corresponding to our system that lie between 0 and 5.
Let us consider the case where $\alpha = 100\beta$ to eliminate the cosine term and to observe the variation as the condition varies between the two values .
Solving this MATLAB, we obtain
The value of α is 0.28 and the value of β is 0.0028

The complex-order fractional differental equation becomes

$$ D^{0.28+0.0028i} x = -ex - 2\,e\sqrt{456.36 U_\Phi} \quad (8) $$

- **Model of $\dot{x}_4 = -1.4540 \times 10^{-1} x_4^2 + 65.295 U_\theta$**

The solution to this differential equation in term of U_Φ for the initial condition x_4=0 is

$$x_4 = \sqrt{449.07U_\theta}\frac{1-\exp(-6.1624\sqrt{U_\theta}t)}{1+\exp(-6.1624\sqrt{U_\theta}t)} \quad (9)$$

Let us use the Taylor series expansion of x_4 which gives

$$x_4(t) = x_4(t) = 261U_\theta t + 1609U_\theta^{\frac{3}{2}}t^2 + 9918{U_\theta}^2t^3 + \ldots \quad (10)$$

The formula for the complex-order fractional integrator which a right-hand side is given by

$$D^{Re+iIm}x = ax + 2\,a \times \sqrt{449.07U_\theta} \quad (11)$$

Using the fractional Taylor series expansion of the solution to this differential equation, we obtain [6][7].

$$X=2 \times \sqrt{449.07U_\theta}\left(\frac{\sqrt[\alpha]{at}}{1!} + \frac{\left(\sqrt[\alpha]{at}\right)^2}{2!} + \frac{\left(\sqrt[\alpha]{at}\right)^3}{3!} + \ldots\right) \quad (12)$$

After simplifying $a^{\frac{-i\beta}{\alpha^2+\beta^2}}$, we obtain

$$a^{\frac{-i\beta}{\alpha^2+\beta^2}} = \exp\left(\ln\left(a^{\frac{-i\beta}{\alpha^2+\beta^2}}\right)\right) \quad (13)$$

Where α is the real part of the fractional model and β is the imaginary part of the fractional model

By comparing the two Taylor series expansions and using the previous equality, we obtain

$$a^{\frac{\alpha}{\alpha^2+\beta^2}}\cos\left(\frac{\beta}{\alpha^2+\beta^2}\ln a\right) = 6.1624\sqrt{U_\theta} \quad (14)$$

According to this relationship, Re and Im depend on the input $\sqrt{U_\theta}$ but let us choose input values corresponding to our system that lie between 0 and 5.
Let us consider the case where $\alpha = 100\beta$ to eliminate the cosine term and to observe the variation as the condition change between the two values/.
Solving this in MATLAB, we obtain.
The value of α is 0.28 and the value of β is0.0028

The complex-order fractional differential equation becomes

$$D^{0.28+0.0028i}x = -ex - 2\,e\sqrt{449.07U_\theta} \quad (15)$$

- **Model of $\dot{x}_6 = -8.2987 \times 10^{-4}x_6^2 + 32.65U_\psi$**

The solution to this differential equation in terms of U_Φ for the initial condition x_4=0 is

$$x_6 = \sqrt{3.934 \times 10^4 U_\psi}\frac{1-\exp(-0.3292\sqrt{U_\psi}t)}{1+\exp(-0.3292\sqrt{U_\psi}t)} \quad (16)$$

Let us use the Taylor series expansion of x_4 which gives

$$x_6(t) = 131U_\psi t + 43\boldsymbol{U_\psi}^{\frac{3}{2}}t^2 + 14{U_\psi}^2t^3 + \ldots \quad (17)$$

The formula for the complex-order fractional integrator which a right-hand side is given by

$$D^{\propto}x = ax + 2\,a \times \sqrt{3.934 \times 10^4 U_\psi} \quad (18)$$

Using the fractional Taylor series expansion of the solution to this differential equation, we obtain [6][7]

$$X=2 \times \sqrt{3.934 \times 10^4 U_\psi}\left(\frac{\sqrt[\alpha]{at}}{1!} + \frac{\left(\sqrt[\alpha]{at}\right)^2}{2!} + \frac{\left(\sqrt[\alpha]{at}\right)^3}{3!} + \ldots\right) \quad (19)$$

After simplifying $a^{\frac{-i\beta}{\alpha^2+\beta^2}}$, we obtain

$$a^{\frac{-i\beta}{\alpha^2+\beta^2}} = \exp\left(\ln\left(a^{\frac{-i\beta}{\alpha^2+\beta^2}}\right)\right) \quad (20)$$

Where α is the real part of the fractional model and β the imaginary part of the fractional model.

By comparing the two Taylor series expansions and using the previous equality, we obtain

$$\overset{\propto}{a^{\alpha^2+\beta^2}} \cos\left(\frac{\beta}{\alpha^2+\beta^2}\ln a\right) = 0.3292\sqrt{U_\psi} \tag{21}$$

According to this relationship, Re and Im depend on the input $\sqrt{U_\theta}$ but let us choose input values corresponding to our system that lie between 0 and 5.
Let take the case where $\alpha = 100\beta$ to eliminate the cosine and to see how the relsult varies between the two values
Solving in MATLAB, we obtain
The value of α is 0.75 and the value of β is 0.0075

The complex-order fractional differential equation becomes

$$D^{0.75+0.0075i}x = -ex - 2\,\mathrm{e}\sqrt{3.934 \times 10^4 U_\psi} \tag{22}$$

III. Results and interpretation

To demonstrate the accuracy of the fractional model, let us plot the results of the three fractional models.
There are an infinite number of possible models of the same system using complex-order fractional modelling. However, the main objective is to able to control the system. It is therefore the shape of the curve that interests us here. It is for this reason that the fractional Taylor series expansion was used, in order to facilitate the subsequent control of the system. When a constant input is applied to the first model in this case, a unit input, the output x_2 exhibits the following corresponding variation.

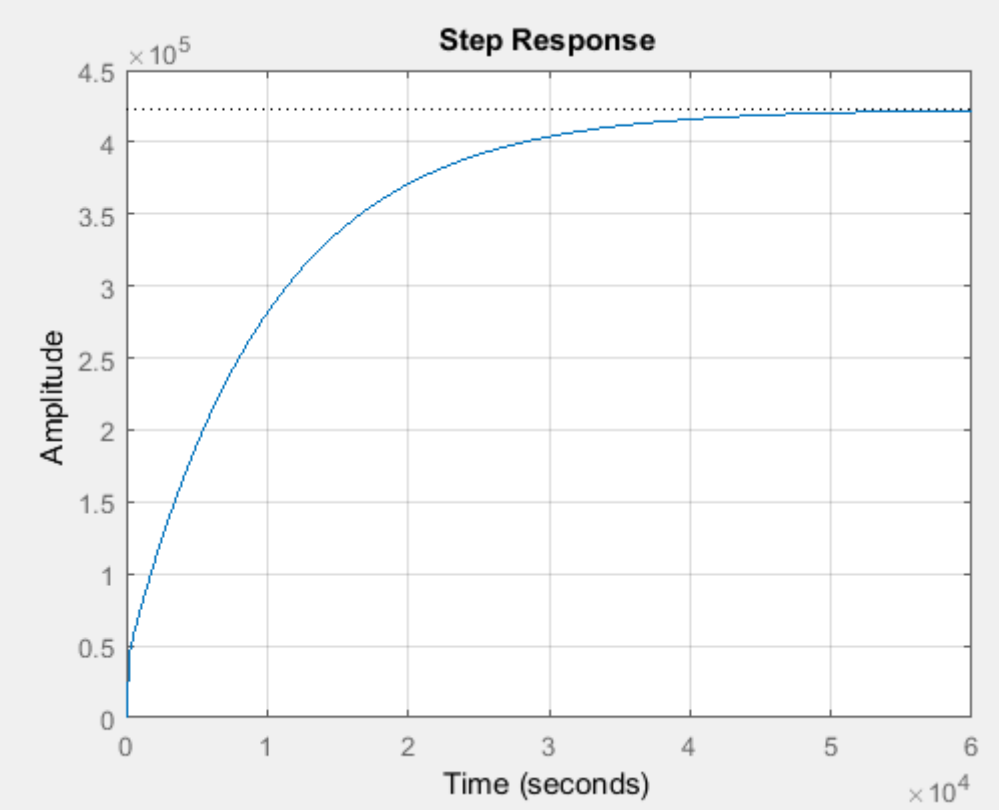


Figure 1: Response of x_2 to a constant input

Comparing the fractional model with the exact model, we observe that the values obtained with the fractional model are much higher. Indeed, we have used the fractional Taylor series expansion, which places the main emphasis on the shape of the curve and the initial conditions, as these are the aspects of interest in this study. Furthermore, if we consider the variation x_2 presented by the equation shown above, the curve rises before tending towards a constant value. The results obtained here exhibit exactly the same behavior.

For the second model, our objective remains unchandged: it is still to make the system controllable. Consequently, the expected results remain the same, namely the preservation of the initial conditions and the shape of the curve. When a constant input is applied to x_4, the following behavior is obtained.

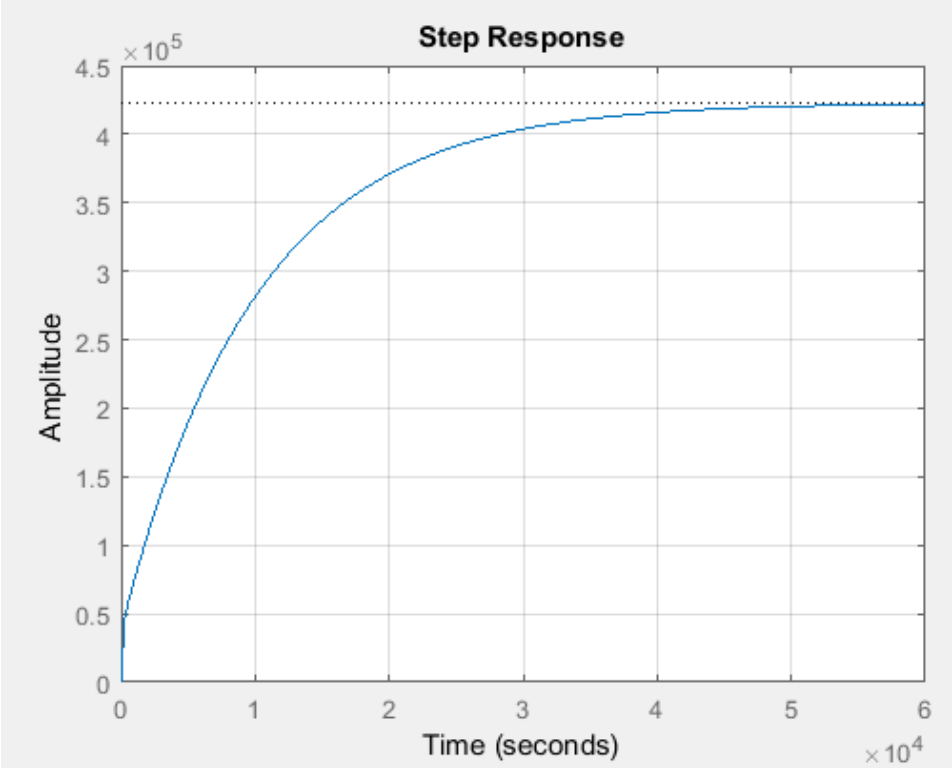


Figure 2: step response of x_4

As shown in the figure, we observe the same behaviour as before : the shape of the resulting curve corresponds perfectly to the expected result, as it rises and then converges to a constant value. Thus, the proposed model is validated and has enabled us to achieve the desired behavior for controlling the system

When we apply a constant input to the third model, under the same conditions as for the first two models, we obtain the following result

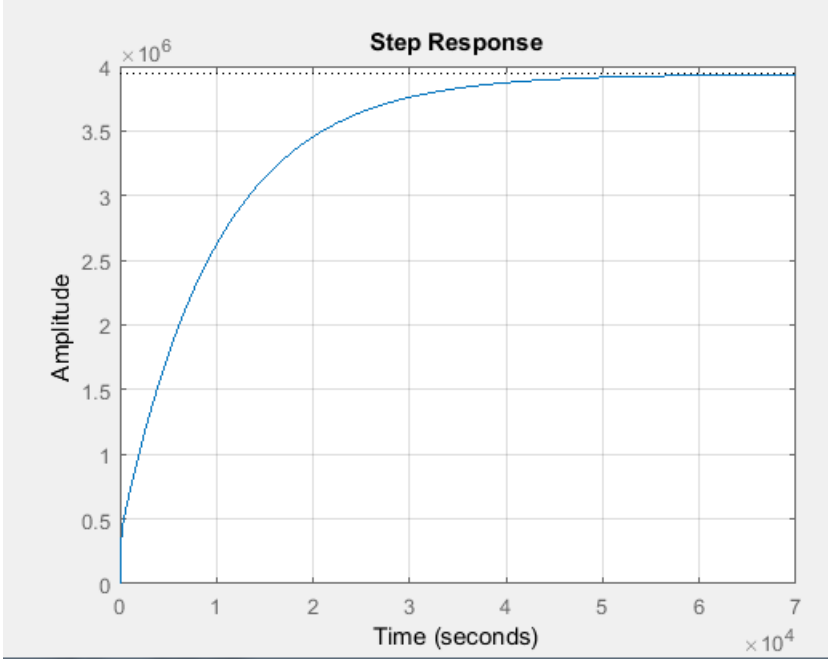


Figure 3:step response of x_6

Here, we can say that the shape obtained is exactly similar to that of the step response of x_6 albeit with higher values. This is because the aim is not to compare the values, but rather to verify the similarity of the response's shape.

IV. Discussion

The CRONE controller is a robust non-integer-order controller. Here, the objective of our fractional-order model is to be able to apply the CRONE controller. Non-linear systems, such as those considered previously, are not suited to the CRONE controller nor, generally speaking, to classical or non-classical linear controllers, expect in the vicinity of the operationg point. [8]

However, the classical linear model used in most control system is based on a first-order Taylor series expansion, which results in a significant loss of information regarding the system's actual dynamics. For example, the previous x_2 becomes simply

$$\dot{x}_2 = 65.51U_\Phi$$

And the exact model no longer takes the same form. Contreversely, the fractional-order model used here retains a more accurate representation of the system by drawing on a more precise model, which is then adapted for use within the CRONE control framework.

Furthermore, in this work, we have used a complex-order fractional model. We could have used a real-order fractional model, since the assumption that $\alpha=100\beta$ is a merely a conjecture.

However, as previously discussed, the use of a complex-order fractional model allows for the modification of a greater number of modeling parameters for modeling. In the case of a real-order fractional model, although it possesses more degrees of freedom for modeling than a conventional integer model, these degrees of freedom remain very limited compared to a complex-order fractional model, which has an infinite number of possible values thanks to the combination of the real and imaginary parts in the form Re+iIm. Finally, the third-generation CRONE controller uses complex-order operators. Consequently, complex-order fractional modeling is best suited to this approach. The use of a second-generation CRONE controller with the real-order fractional model remains possible, but it is less robust than the first-generation fractional-order PID controller.

V. Conclusion

Complex-order fractional modeling takes the same form as real-order fractional modeling but with complex coefficients. This type of modeling is difficult to design, but it offers a very high degree of modeling flexibility. We have approximated the exact drone model using this type of model, and the resulting model is very simple. Consequently, we have demonstrated that the fractional model can be easily controlled by CRONE. All the conditions initially imposed for the tree simulated models were met. We can therefore conclude that the proposed model is effective for non-linear differential equations. Now, an important question arises: what would the fractional model be if the non-linear system cannot be written in the form of a non-linear differential equation?

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