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# A NOTE OF NONINVARIANT HYPERSURFACES OF PARA SASAKIAN MANIFOLD

**Dhruwa Narain[1], Sachin Kumar Srivastava[2] and Khushbu Srivastava[3]**
[1,3]Department of Mathematics & Statistics
D.D.U. Gorakhpur University, Gorakhpur, INDIA
[2]Department of Applied Sciences and Humanities
Babu Banarasi Das Institute of Technology, Ghaziabad, INDIA

**ABSTRACT:**

In 1970, S.I. Goldberg and K. Yano introduced the notion of noninvariant hypersurface $\left(\tilde{M}\right)$.The present paper deals with the properties of noninvariant hypersurfaces of para Sasakian manifold.Some theorems are obtained.
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## 1. INTRODUCTION:

An n-dimensional differentiable manifold M is called an almost para contact manifold if it admits an almost para contact structure

$(\phi, \xi, \eta)$ consisting of a (1,1) tensor field $\phi$, a vector field $\xi$, and a 1-form $\eta$ satisfying:

**(1.1)** $\phi^2 = Id - \eta \otimes \xi$,

**(1.2)** $\eta(\xi) = 1$,

**(1.3)** $\eta \circ \phi = 0$,

**(1.4)** $\phi \xi = 0$.

Let g be a compatible Riemannian metric with $(\phi, \xi, \eta)$, i.e.,

**(1.5)** $g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y)$

or equivalently,

**(1.6)** $g(X, \phi Y) = g(\phi X, Y)$ and $g(X, \xi) = \eta(X)$

for all $X, Y \in TM$.

Then M becomes an almost para contact Riemannian manifold equipped with an almost para contact Riemannian structure $(\phi, \xi, \eta, g)$.

An almost para contact Riemannian manifold is called a p-Sasakian manifold if it satisfies:

**(1.7)** $(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi$ ; $X, Y \in TM$

where $\nabla$ is Levi-Civita connection of the Riemannian metric. From the above equation it follows that:

**(1.8)** $\nabla_\xi X = \phi X$,

**(1.9)** $(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X$ ; $X \in TM$.

In an n-dimensional p-Sasakian manifold M, the curvature tensor K, the Ricci tensor R, and the Ricci operator Q satisfy:

**(1.10)** $K(X,Y)\xi = \eta(X)Y - \eta(Y)X$

**(1.11)** $K(\xi, X)Y = \eta(Y)X - g(X,Y)\xi$

**(1.12)** $K(\xi, X)T = X - \eta(X)T$

**(1.13)** $R(X, \xi) = -(n-1)\eta(X)$

**(1.14)** $Q\xi = -(n-1)\xi$

**(1.15)** $\eta(K(X,Y)U) = g(X,U)\eta(Y) - g(Y,U)\eta(X)$

**(1.16)** $\eta(K(X,Y)\xi) = 0$

**(1.17)** $\eta(K(\xi, X)Y) = \eta(X)\eta(Y) - g(X,Y)$

An almost para contact Riemannian manifold M is said to be $\eta$-Einstein [2] if the Ricci operator Q satisfies:

**(1.18)** $Q = a\,Id + b\,\eta \otimes \xi$

where a and b are smooth functions on the manifold. In particular, if b=0, then M is an Einstein manifold.

Let (M,g) be an n-dimensional Riemannian manifold. Then the concircular curvature tensor C and the Weyl conformal curvature tensor W are defiend by [2]:

**(1.19)** $C(X,Y)U = K(X,Y)U - \frac{r}{n(n-1)}\{g(Y,U)X - g(X,U)Y\}$

**(1.20)** $W(X,Y)U = K(X,Y)U - \frac{1}{(n-2)}\{R(Y,U)X - R(X,U)Y + g(Y,U)QX - g(X,U)QY\} + \frac{r}{(n-1)(n-2)}\{g(Y,U)X - g(X,U)Y\}$

for all X, Y,U $\in$ TM , respectively, where r is the scalar curvature of M.

## 2. HYPERSURFACE OF PARA SASAKIAN MANIFOLDS:

Let $M_n$ be an n-dimensional Riemannian manifold with positive definite metric g and let $M_{n-1}$ be a hypersurface immersed in $M_n$.

If $i_*$ denotes the differential of the immersion i of $M_{n-1}$ into $M_n$ and $\tilde{X}$ is a vector field on $M_{n-1}$.

Let N be the unit normal field to $M_{n-1}$. The induced metric $\tilde{g}$ on $M_{n-1}$ is defiend by:

**(2.1)** $\tilde{g}(\tilde{X}, \tilde{Y}) = g(\tilde{X}, \tilde{Y})$

we have,

**(2.2)** $\tilde{g}(\tilde{X}, N) = 0$ and $\tilde{g}(N, N) = 1$ .

If $\nabla$ is the Riemannian connection in $M_n$, then the Gauss and Weingarten formulae are given respectively by:

**(2.3)** $$\nabla_{\tilde{X}}\tilde{Y} = \tilde{\nabla}_{\tilde{X}}\tilde{Y} + h\left(\tilde{X},\tilde{Y}\right)N$$

**(2.4)** $$\nabla_{\tilde{X}}N = -H\left(\tilde{X}\right)$$

where $\tilde{\nabla}$ is the induced Riemannian connection in $M_{n-1}$ and h is the second fundamental tensor satisfying:

**(2.5)** $$h(\tilde{X}, \tilde{Y}) = h(\tilde{Y}, \tilde{X}) = \tilde{g}(H(\tilde{X}), \tilde{Y})$$

***Remark***:- *On all objects of $M_{n-1}$ will be denoted with hyphen '~' placed over*

*them **eg**. $\tilde{\phi}$, $\tilde{X}$ etc.*

Now suppose that ($\phi$, $\xi$, $\eta$, g) is an almost para contact Riemannian structure on $M_n$. Then every vector field X on $M_n$ is decomposed as:

**(2.6)** $$X = \tilde{X} + \alpha(X)N$$

where $\alpha$ is a 1-form on $M_n$ , and for every vector field $\tilde{X}$ on $M_{n-1}$ and the normal N , we have

**(2.7)** $$\phi\tilde{X} = \tilde{\phi}\tilde{X} + \beta\left(\tilde{X}\right)N$$

**(2.8)** $$\phi N = \tilde{N} + \lambda\left(N\right)$$

where f is a tensor field of type (1, 1) on a hypersurface $M_{n-1}$ , $\beta$ is a 1-form on $M_{n-1}$ and $\lambda$ is a scalar function on $M_{n-1}$ .

If $\beta \neq 0$ , we call $M_{n-1}$ a non-invariant hypersurface of M.

Now, we have suppose that

**(2.9)** $$\xi = \tilde{\xi} + \gamma N$$

where,

**(2.10)** $$g\left(\tilde{\xi},\tilde{X}\right) = \tilde{\eta}\left(\tilde{X}\right) = \eta\left(\tilde{X}\right)$$ , $\gamma$ is a scalar function.

From (2.7), (2.8), (2.9) and (2.10), we have

**(2.11)** $$\tilde{\phi}^2\tilde{X} + \beta\left(\tilde{X}\right)\tilde{N} = X - \eta\left(\tilde{X}\right)\tilde{\xi}$$

**(2.12)** $$\beta\left(\tilde{\phi}\tilde{X}\right) + \lambda\beta\left(\tilde{X}\right) = -\gamma\eta\left(\tilde{X}\right)$$

**(2.13)** $\beta(\tilde{N}) + \gamma\eta(N) = 1 - \lambda^2$

**(2.14)** $\tilde{\phi}\tilde{N} = -\tilde{\xi}\eta(N) - \lambda\tilde{N}$

**(2.15)** $\tilde{\phi}\tilde{\xi} + \gamma\tilde{N} = 0$

**(2.16)** $\beta(\tilde{\xi}) + \lambda\gamma = 0$

**(2.17)** $(\tilde{\xi} \circ \tilde{\phi})\tilde{X} + \beta(\tilde{X})\eta(N) = 0$

Since,

$$g(\phi\tilde{X}, \phi\tilde{Y}) = g(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y})$$

$$\Rightarrow \quad g(\tilde{\phi}\tilde{X} + \beta(\tilde{X})N, \tilde{\phi}\tilde{Y} + \beta(\tilde{Y})N) = g(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y})$$

$$\Rightarrow \quad g(\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{Y}) = g(\tilde{X}, \tilde{Y}) - \eta(\tilde{X})\eta(\tilde{Y}) - \beta(\tilde{X})\beta(\tilde{Y})$$

i.e.

**(2.18)** $g(\tilde{\phi}\tilde{X}, \tilde{\phi}\tilde{Y}) = g(\tilde{X}, \tilde{Y}) - \tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}) - \beta(\tilde{X})\beta(\tilde{Y})$

Let $'F(\tilde{X}, \tilde{Y}) = g(\phi\tilde{X}, \tilde{Y})$

then $$'F(\tilde{X}, \tilde{Y}) = g(\tilde{\phi}\tilde{X} + \beta(\tilde{X})N, \tilde{Y}) = g(\tilde{\phi}\tilde{X}, \tilde{Y})$$

i.e.

**(2.19)** $'F(\tilde{X}, \tilde{Y}) = '\tilde{F}(\tilde{X}, \tilde{Y})$

Since ,

$$0 = g(\tilde{\phi}\tilde{X}, N) = g(\phi\tilde{X} - \beta(\tilde{X})N, N)$$

$$= g(\phi\tilde{X}, N) - \beta(\tilde{X})$$

$$= g(\tilde{X}, \phi N) - \beta(\tilde{X})$$

$$= g(\tilde{X}, \tilde{N} + \lambda N) - \beta(\tilde{X})$$

$$= g(\tilde{X}, \tilde{N}) - \beta(\tilde{X})$$

i.e. ,



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**(2.20)** $$g(\tilde{X},\tilde{N}) = \beta(\tilde{X})\,\tilde{g}(\tilde{X},\tilde{N})$$

Differentiating covariantly (2.7), and (2.8) along $M_{n-1}$ , using (2.3), (2.4), and (2.20), we have

**(2.21)**
$$(\nabla_{\tilde{Y}}\phi)(\tilde{X}) = (\tilde{\nabla}_{\tilde{Y}}\tilde{\phi})(\tilde{X}) + \{(\tilde{\nabla}_{\tilde{Y}}\beta)(\tilde{X}) - \lambda h(\tilde{X},\tilde{Y}) + h(\tilde{\phi}\tilde{X},\tilde{Y})\}N - h(\tilde{X},\tilde{Y})\tilde{N} + \beta(\tilde{X})\{\tilde{\nabla}_{\tilde{Y}}N + h(N,\tilde{Y})N\}$$

and

**(2.22)** $$(\nabla_{\tilde{Y}}\phi)N = (\tilde{\nabla}_{\tilde{Y}}\tilde{N}) + \tilde{\phi}(H(\tilde{Y})) - \lambda H(\tilde{Y}) + \{2h(\tilde{Y},\tilde{N}) + \tilde{Y}(\lambda)\}N$$

From (2.9) we have ,

**(2.23)** $$\nabla_{\tilde{Y}}\xi = \nabla_{\tilde{Y}}\tilde{\xi} + \{h(\tilde{Y},\tilde{\xi}) + \tilde{Y}(\gamma)\}N - H(\tilde{Y})\gamma$$

If almost para – Sasakian manifold satisfying ( $\phi$ , $\xi$ , $\eta$ )- connection
i.e. $\nabla\phi = 0,\ \nabla\xi = 0,\ \nabla\eta = 0$ where $\nabla$ denotes covariant differentiation with respect to a symmetric affine connection on $M_n$ .

Hence, we can state the following theorem:

**Theorem (2.1):** On the non-invariant hypersurface of almost para –Sasakian manifold with ( $\phi$ , $\xi$ , $\eta$ ) connection, we have

(a) $(\tilde{\nabla}_{\tilde{X}}\tilde{\phi})\tilde{Y} = h(\tilde{X},\tilde{Y})\tilde{N} + H(\tilde{X})\beta(\tilde{Y}) - \{(\tilde{\nabla}_{\tilde{X}}\beta)\tilde{Y} - \lambda h(\tilde{X},\tilde{Y}) + \lambda h(\tilde{X},\tilde{\phi}\tilde{Y})\}N$

(b) $(\tilde{\nabla}_{\tilde{X}}\tilde{N})\tilde{Y} = \lambda H(\tilde{X}) - \tilde{\phi}(H(\tilde{X})) - \{2h(\tilde{X},\tilde{N}) + \tilde{X}(\lambda)\}N$

(c) $(\tilde{\nabla}_{\tilde{Y}}\tilde{\xi}) = H(\tilde{Y})\gamma - \{h(\tilde{Y},\tilde{\xi}) + \tilde{Y}(\gamma)\}N$

(d) $(\tilde{\nabla}_{\tilde{Y}}\tilde{\eta})\tilde{X} = \gamma h(\tilde{X},\tilde{Y})$

We know that an almost para contact manifold is called p-Sasakian manifold if it satisfies:

**(2.24)** $$(\tilde{\nabla}_{\tilde{X}}\ \phi)\tilde{Y} = -\,g(\tilde{X},\tilde{Y})\ \xi\ -\ \eta\ (\tilde{Y})\ \tilde{X}\ \mathrm{X} + 2\eta\ (\tilde{X})\ \eta\ (\tilde{Y})\ \xi$$

using (2.24) in equation (2.21), we have

$$(\tilde{\nabla}_{\tilde{X}}\tilde{\phi})\tilde{Y} - h(\tilde{X},\tilde{Y})\tilde{N} + \beta(\tilde{Y})\tilde{\nabla}_{\tilde{X}}N = -g(\tilde{X},\tilde{Y})\tilde{\xi} - \eta(\tilde{Y})\tilde{X} + 2\eta(\tilde{X})\eta(\tilde{Y})\tilde{\xi}$$

and

$$(\tilde{\nabla}_{\tilde{X}}\beta)\tilde{Y} - \lambda h(\tilde{X},\tilde{Y}) + h(\tilde{X},\tilde{\phi}\tilde{Y}) + \beta(\tilde{Y})h(N,\tilde{X}) = -\lambda\left(g(\tilde{X},\tilde{Y})\tilde{\xi} - 2\eta(\tilde{X})\eta(\tilde{Y})\right)$$

This leads to the following theorem:

**Theorem (2.2):** On the non-invariant hypersurface of a p-Sasakian manifold, we have

IOSRJEN

(a) $\left(\tilde{\nabla}_{\tilde{X}}\tilde{\phi}\right)\tilde{Y}-h\left(\tilde{X},\tilde{Y}\right)\tilde{N}+\beta\left(\tilde{Y}\right)\tilde{\nabla}_{\tilde{X}}N=-g\left(\tilde{X},\tilde{Y}\right)\tilde{\xi}-\eta\left(\tilde{Y}\right)\tilde{X}+2\eta\left(\tilde{X}\right)\eta\left(\tilde{Y}\right)\tilde{\xi}$

(b) $\left(\tilde{\nabla}_{\tilde{X}}\beta\right)\tilde{Y}-\lambda h\left(\tilde{X},\tilde{Y}\right)+h\left(\tilde{X},\tilde{\phi}\tilde{Y}\right)+\beta\left(\tilde{Y}\right)h\left(N,\tilde{X}\right)=-\lambda\left(g\left(\tilde{X},\tilde{Y}\right)\tilde{\xi}-2\eta\left(\tilde{X}\right)\eta\left(\tilde{Y}\right)\right)$

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