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Some Results On Product of Preece's Identity and its Generalization

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Abstracts: The aim of this paper is to establish Some Results On Product of generalized hypergeometric Function and its Generalizations of the well known Preece's identity using method in the line of author Rathie Arjun K. and Choi Junesang [8].

I. INTRODUCTION

Due to Rainville E.D.[10], Generalized hypergeometric function with p numerator and q denominator parameters is defined as

$${}_pF_q(a_1,\text{---},a_p\,;\,b_1,\text{---},b_q\,;\,x\,) = \sum_{m,n=0}^{\infty} \frac{(a_1)_n \cdots\cdots (a_p)_n}{(b_1)_n \cdots\cdots (b_q)_n \;\; n!} x^m \qquad \text{----------------} \; (1.1)$$

where $(a)_n$ denotes the Pochhammer symbol defined by

$(a)_0 = 1$ and $(a)_n = a(a+1) \ldots \ldots (a+n-1) \;\; ; \;\; (n = 1,2,3, \ldots)$, for any complex number a.

From the theory of differential equations, Professor Preece C.T.[5] established the following very interesting identity involving product of Generalized hypergeometric series:

$${}_1F_1(a\,;\,2a\,;\,x) \times {}_1F_1(1\text{-}a\,;\,2\text{-}2a\,;\,x) = e^x \; {}_1F_2(\tfrac{1}{2} \;;\, a+\tfrac{1}{2}\,, \tfrac{3}{2} - a\,;\, \tfrac{x^2}{4}) \qquad \text{-----------------} \; (1.2)$$

Rathie Arjun K.[6], has given a very short proof of (1.2). In another paper Rathie Arjun K.[7] has also obtained the following two results contiguous to (1.2):

$$\begin{aligned}{}_1F_1(a\,;\,2a+1\,;\,x) \times {}_1F_1(1\text{-}a\,;\,2\text{-}2a\,;\,x) = e^x \; \{&{}_1F_2(\tfrac{1}{2} \;;\, a+\tfrac{1}{2}\,, \tfrac{3}{2} - a\,;\, \tfrac{x^2}{4}) \\ &- \frac{x}{2(2a+1)}\, {}_2F_3(\tfrac{3}{2}\,, 1\,;\, \tfrac{3}{2} - a\,,\, a+\tfrac{3}{2}\,;\, \tfrac{x^2}{4})\end{aligned} \qquad \text{-----------------} \; (1.3)$$

and

$$\begin{aligned}{}_1F_1(a\,;\,2a - 1\,;\,x) \times {}_1F_1(1 - a\,;\,2 - 2a\,;\,x) = e^x \; \{&{}_1F_2(\tfrac{1}{2} \;;\, a - \tfrac{1}{2}\,, \tfrac{3}{2} - a\,;\, \tfrac{x^2}{4}) \\ &+ \frac{x}{2(2a-1)}\, {}_1F_2(\tfrac{1}{2}\,;\, \tfrac{3}{2} - a\,,\, a + \tfrac{1}{2}\,;\, \tfrac{x^2}{4}).\end{aligned} \qquad \text{-----------------} \; (1.4)$$

We recall the following interesting formula due to Srivastava H. M. and Karlsson P.W.[11] (p. 322):

$${}_1F_1(a\,;\,2a\,;\,x) \times {}_1F_1(b\,;\,2b\,;\,\text{-}x) = {}_2F_3(\tfrac{1}{2}(a+b)\,, \tfrac{1}{2}(a+b+1)\,;\, a+\tfrac{1}{2}\,,\, b+\tfrac{1}{2}\,,\, a+b\,;\, \tfrac{x^2}{4})\,. \; \text{--} \; (1.5)$$

We also recall the following results due to Rathie Arjun K. and Choi Junesang [8] (p. 341)

$$\begin{aligned}&{}_1F_1(a\,;\,2a\,;\,x) \times {}_1F_1(b\,;\,2b\,;\,x) \\ &= e^x \; {}_2F_3(\tfrac{1}{2}(a+b)\,, \tfrac{1}{2}(a+b+1)\,;\, a+\tfrac{1}{2}\,,\, b+\tfrac{1}{2}\,,\, a+b\,;\, \tfrac{x^2}{4})\,,\end{aligned} \qquad \text{---------------} \; (1.6)$$

$${}_1F_1(a\,;\,2a+1\,;\,x) \times {}_1F_1(b\,;\,2b\,;\,x)$$

$= e^x \{{}_2F_3(\frac{1}{2}(a+b+1), \frac{1}{2}(a+b); a+\frac{1}{2}, b+\frac{1}{2}, a+b; \frac{x^2}{4}) - \frac{x}{2(2a+1)}$
$\times {}_2F_3(\frac{1}{2}(a+b+2), \frac{1}{2}(a+b+1); a+\frac{3}{2}, b+\frac{1}{2}, a+b+1; \frac{x^2}{4})\}$ ---------- (1.7)

${}_1F_1(a; 2a-1; x) \times {}_1F_1(b; 2b; x)$
$= e^x \{{}_2F_3(\frac{1}{2}(a+b-1), \frac{1}{2}(a+b); b+\frac{1}{2}, a-\frac{1}{2}, a+b-1; \frac{x^2}{4}) + \frac{x}{2(2a-1)}$
$\times {}_2F_3(\frac{1}{2}(a+b+1), \frac{1}{2}(a+b); b+\frac{1}{2}, a+\frac{1}{2}, a+b; \frac{x^2}{4})\}$. -------- (1.8)

The following results are due to Rathie Arjun K. and Choi Junesang [8] (p. 343):

${}_1F_1(a; 2a; x) \times {}_1F_1(2-a; 4-2a; x) = e^x \, {}_2F_3(1, \frac{3}{2}; a+\frac{1}{2}, \frac{5}{2}-a, 2; \frac{x^2}{4})$ --------- (1.9)

${}_1F_1(a; 2a; x) \times {}_1F_1(3-a; 6-2a; x) = e^x \, {}_2F_3(\frac{3}{2}, 2; a+\frac{1}{2}, \frac{7}{2}-a, 3; \frac{x^2}{4})$. --------- (1.10)

The following is an interesting result due to Bailey W. N.[2]:

${}_0F_1(-; a; x) \times {}_0F_1(-; b; x) = {}_2F_3(\frac{1}{2}(a+b), \frac{1}{2}(a+b-1); a, b, a+b-1; 4x)$. ----------- (1.11)

The well-known Kummer's first and second theorems due to Kummer E. E.[4], are

${}_1F_1(a; c; x) = e^x \, {}_1F_1(c-a; c; -x)$, ------------------ (1.12)

$e^{-x} \, {}_1F_1(a; 2a; x) = {}_0F_1(-; a+\frac{1}{2}; \frac{x^2}{16})$. ------------------ (1.13)

The Rathie Arjun K. and Nagar V. [9] have obtained the following two interesting results contiguous to (1.5):

$e^{-x/2} \, {}_1F_1(a; 2a+1; x) = {}_0F_1(-; a+\frac{1}{2}; \frac{x^2}{16}) - \frac{x}{2(2a+1)} \, {}_0F_1(-; a+\frac{3}{2}; \frac{x^2}{16})$, ------------- (1.14)

and

$e^{-x/2} \, {}_1F_1(a; 2a-1; x) = {}_0F_1(-; a-\frac{1}{2}; \frac{x^2}{16}) + \frac{x}{2(2a-1)} \, {}_0F_1(-; a+\frac{1}{2}; \frac{x^2}{16})$. ------------ (1.15)

The aim of this paper is to establish Some Results On Product of Generalized hypergeometric Function and its Generalizations of the well known Preece's identity (1.2) by a similar method given by Rathie Arjun K.[6] & Rathie Arjun K. and Choi Junesang [8].

II. MAIN RESULTS

In this section, we shall establish the following results on product of Generalized hypergeometric function and its Generalizations of the well known Preece's identity (1.2):

${}_1F_1(a; 2a; 2x) \times {}_1F_1(a; 2a; -2x) = e^{2x} \, {}_2F_3(a+\frac{1}{2}, a; a+\frac{1}{2}, a+\frac{1}{2}, 2a; x^2)$ ------ (2.1)

${}_1F_1(a; 2a+1; 2x) \times {}_1F_1(a; 2a+1; 2x)$

$= e^{2x} [{}_2F_3(a+\frac{1}{2}, a; a+\frac{1}{2}, a+\frac{1}{2}, 2a; x^2)$
$+ \frac{x^2}{(2a+1)^2} \, {}_2F_3(a+\frac{3}{2}, a+1; a+\frac{3}{2}, a+\frac{3}{2}, 2a+2; x^2)$
$- \frac{2x}{(2a+1)} \, {}_2F_3(a+1, a+\frac{1}{2}; a+\frac{1}{2}, a+\frac{3}{2}, 2a+1; x^2)]$ ----- (2.2)

${}_1F_1(a; 2a-1; 2x) \times {}_1F_1(a; 2a-1; 2x)$
$= e^{2x} [{}_2F_3(a-\frac{1}{2}, a-1; a-\frac{1}{2}, a-\frac{1}{2}, 2a-2; x^2)$

$$+ \frac{x^2}{(2a-1)^2} \, {}_2F_3(a+\tfrac{1}{2}, a \,;\, a+\tfrac{1}{2}, a+\tfrac{1}{2}, 2a \,;\, x^2)$$
$$+ \frac{2x}{(2a-1)} \, {}_2F_3(a, a-\tfrac{1}{2} \,;\, a-\tfrac{1}{2}, a+\tfrac{1}{2}, 2a-1 \,;\, x^2)] \qquad \text{----- (2.3)}$$

Proof of (2.1) :-
Using equation (1.12), we get

$${}_1F_1(a\,;2a\,;2x) \times {}_1F_1(a\,;2a\,;-2x) = {}_1F_1(a\,;2a\,;2x) \times e^{-2x}\,{}_1F_1(a\,;2a\,;2x)$$

Using equation (1.13), we have

$$= e^{2x}\,{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4}) \times e^{-2x}\, e^{2x}\,{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4})$$
$$= e^{2x}\,{}_2F_3(a+\tfrac{1}{2}, a\,;a+\tfrac{1}{2}, a+\tfrac{1}{2}, 2a\,;x^2) \qquad \text{---- (2.4)}$$

In this way (2.1) is proved.

Proof of (2.2) :-
Using equation (1.13) & equation (1.14), we get

$${}_1F_1(a\,;2a+1\,;2x) \times {}_1F_1(a\,;2a+1\,;2x)$$

$$= e^{x}\,[{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4}) - \frac{x}{(2a+1)}\,{}_0F_1(-\,;a+\tfrac{3}{2}\,;\tfrac{x^2}{4})]$$
$$\times e^{x}\,[{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4}) - \frac{x}{(2a+1)}\,{}_0F_1(-\,;a+\tfrac{3}{2}\,;\tfrac{x^2}{4})]$$

$$= e^{2x}[{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4}) \times {}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4})$$
$$+ \frac{x^2}{(2a+1)^2}\,{}_0F_1(-\,;a+\tfrac{3}{2}\,;\tfrac{x^2}{4}) \times {}_0F_1(-\,;a+\tfrac{3}{2}\,;\tfrac{x^2}{4})$$
$$- \frac{2x}{(2a+1)}\,{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4}) \times {}_0F_1(-\,;a+\tfrac{3}{2}\,;\tfrac{x^2}{4})]$$

Using equation (1.11), we have

$$= e^{2x}\,[{}_2F_3(a+\tfrac{1}{2}, a\,;a+\tfrac{1}{2}, a+\tfrac{1}{2}, 2a\,;x^2)$$
$$+ \frac{x^2}{(2a+1)^2}\,{}_2F_3(a+\tfrac{3}{2}, a+1\,;a+\tfrac{3}{2}, a+\tfrac{3}{2}, 2a+2\,;x^2)$$
$$- \frac{2x}{(2a+1)}\,{}_2F_3(a+1, a+\tfrac{1}{2}\,;a+\tfrac{1}{2}, a+\tfrac{3}{2}, 2a+1\,;x^2)] \qquad \text{---- (2.5)}$$

In this way (2.2) is proved.

Proof of (2.3) :-
Using equation (1.13) & equation (1.15), we get

$${}_1F_1(a\,;2a-1\,;2x) \times {}_1F_1(a\,;2a-1\,;2x)$$

$$= e^{x}\,[{}_0F_1(-\,;a-\tfrac{1}{2}\,;\tfrac{x^2}{4}) + \frac{x}{(2a-1)}\,{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4})]$$
$$\times e^{x}\,[{}_0F_1(-\,;a-\tfrac{1}{2}\,;\tfrac{x^2}{4}) + \frac{x}{(2a-1)}\,{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4})]$$

$$= e^{2x}[{}_0F_1(-\,;a-\tfrac{1}{2}\,;\tfrac{x^2}{4}) \times {}_0F_1(-\,;a-\tfrac{1}{2}\,;\tfrac{x^2}{4})$$
$$+ \frac{x^2}{(2a-1)^2}\,{}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4}) \times {}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4})$$
$$+ \frac{2x}{(2a-1)}\,{}_0F_1(-\,;a-\tfrac{1}{2}\,;\tfrac{x^2}{4}) \times {}_0F_1(-\,;a+\tfrac{1}{2}\,;\tfrac{x^2}{4})]$$

Using equation (1.11), we have

$$= e^{2x}\,[\,{}_2F_3(a - \tfrac{1}{2}, a\text{-}1\ ;\ a - \tfrac{1}{2}\ ,\ a - \tfrac{1}{2}\ ,\ 2a\text{-}2\ ;\ x^2)$$
$$+ \frac{x^2}{(2a-1)^2}\,{}_2F_3(a + \tfrac{1}{2}, a\ ;\ a + \tfrac{1}{2}\ ,\ a + \tfrac{1}{2}\ ,\ 2a\ ;\ x^2)$$
$$+ \frac{2x}{(2a-1)}\,{}_2F_3(a,\ a - \tfrac{1}{2}\ ;\ a - \tfrac{1}{2}, a + \tfrac{1}{2}, 2a\text{-}1\ ;\ x^2\ )] \qquad \text{----- (2.6)}$$

In this way (2.3) is proved.

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REFERENCES

[1] Bailey W.N., "Generalized Hypergeometric Series" (1964).
[2] Bailey W. N., "Products of generalized hypergeometric series", Proc. London Math. Soc., 28 (1928), pp. 242-254.
[3] Erd´elyi A., "*Higher transcendental functions",* McGraw Hill, New York, vol-1 (1953).
[4] Kummer E. E., "Uber die hypergeometrische Reihe F(a; b; x)", J. Reine Angew Math. 15 (1836), pp.39-83.
[5] Preece C. T., "The product of two generalized hypergeometric functions", Proc. London Math. Soc., 22 (1924), pp.370-380.
[6] Rathie Arjun K., "A short proof of Preece's identities and other contiguous results", Rev. Mat. Estatis.
[7] Rathie Arjun K., "On two results contiguous to Preece's identity", (1997).
[8] Rathie Arjun K. and Choi Junesang, "A note on generalizations of preece's identity and other contiguous results", Bull. Korean Math. Soc. 35 (1998), No. 2, pp. 339-344.
[9] Rathie Arjun K. and Nagar V., "On Kummer's second theorem involving product of generalized hypergeometric series", Le Matematiche (catania) 50 (1995),pp. 35-38.
[10] Rainville E.D., "**Special Function"**, Chelsea publication company new yprk,(1960).
[11] Srivastava H. M. and Karlsson P.W.," Multiple Gaussian Hypergeometric Series, Halstead Press", New York, (1985).