IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 05, Issue 04 (April. 2015), ||V3|| PP 46-58

Study of predator switching in an eco-epidemiological model with disease in the prey

Ala'a Abbas M.Rasheed[1] , Raid Kamel Naji[2], Basim N.Abood[1]
[1.] *Department of mathematics, College of science, University Technology, Baghdad, Iraq.*
[2.] *Department of mathematics, College of science, University Baghdad, Baghdad, Iraq.*

Abstract: - In this paper, the dynamical behavior of some eco-epidemiological models is investigated. Prey-predator models involving infectious disease in prey population, which divided it into two compartments; namely susceptible population *S* and infected population *I*, are proposed and analyzed. The proposed model deals with SIS infectious disease that transmitted directly from external sources, as well as, through direct contact between susceptible and infected individuals. in addition to the infectious disease in a prey species, predator switching among susceptible and infected prey population . The model are represented mathematically by the set of nonlinear differential equations .The existence, uniqueness and boundedness of this model are investigated. The local and global stability conditions of all possible equilibrium points are established. Finally, using numerical simulations to study the global dynamics of the model .

Keyword*: prey-predator model; local stability; global stability; switching.*

I. INTRODUCTION

A prey-predator model inovlving predator switching is received a lot of attention in literatures. It is well known that in nature there are different factors; such as stage structure, disease, delay, harvesting and switching; effect the dynamical behavior of the model. In addition, the existence of some type of disease divides the population into two classes known as susceptible and infected. Further, since the infected prey is weaker than the susceptible prey therefore catching the infected prey by a predator will be easer than catching the susceptible prey. Consequently most of the prey-predator models assume that the predator prefers to eat the infected prey, however when the infected prey species is rare or in significant defense capability with respect to predation then a predator switches to feed on the susceptible prey species. Number of mathematical models involving predator switching due to various reasons have been studied. Tansky [6]investigated a mathematical model of one predator and two prey system where the predator has switching tendency. Khan [3]studied a prey- predator model with predator switching where the prey species have the group defense capability. Malchow [4]studied an excitable plankton ecosystem model with lysogenic viral infection that incorporates predator switching using a Holling type-III functional response.

Keeping the above in view, in this paper, a prey-predator model involving SIS infectious disease in prey species with predator switching is proposed and analyzed. It is assumed that the disease transmitted within the prey population by direct contact and though an external sources. The existence, uniqueness and boundedness of the solution are discussed. The existence and the stability analysis of all possible equilibrium points are studied. Finally, the global dynamics of the model is carried out analytically as well as numerically.

The mathematical model:

In this section an eco-epidemiological model describing a prey-predator system with switching in the predator is proposed for study, the system involving an *SIS* epidemic disease in prey population. In the presence of disease, the prey population is divided into two classes: the susceptible individuals $S(t)$ and the infected individuals $I(t)$, here $S(t)$ represents the density of susceptible individuals at time *t* while $I(t)$ represents the infected individuals at time *t*. The prey population grows logistically with intrinsic growth rate $r > 0$ and environmental carrying capacity $K > 0$. The existence of disease causes death in the infected prey with positive death rate $d_1 > 0$. The predator species consumes the prey species (susceptible as well as infected) according to predation rate represented by the switching functions [5]with switching rate constants $P_1 > 0$ and $P_2 > 0$ respectively, however it converts the food from susceptible and infected prey with a conversion rates $e_1 > 0$ and $e_2 > 0$ respectively. Finally, in the absence of prey species the predator species decay exponentially with a natural death rate $d_2 > 0$. Now in order to formulate our model, the following assumptions are adopted:

Consider a prey-predator system in which the density of prey at time t is denoted by $N(t)$ while the density of predator species at time t is denoted by $Y(t)$. Let the following assumptions are adopted:

1. Only the susceptible prey can reproduce logistically, however the infected prey can't reproduce but still has a capability to compete with the other prey individuals for carrying capacity.
2. The susceptible prey becomes infected prey due to contact between both the species as well as external sources for the infection with the contact infection rate constant $\beta > 0$ and an external infection rate $C > 0$. However, the infected prey recover and return to the susceptible prey with a recover rate constant $\gamma > 0$.
3. The functions $\frac{P_1 SY}{1+\left(\frac{I}{S}\right)^2}$ and $\frac{P_2 IY}{1+\left(\frac{S}{I}\right)^2}$ mathematically characterize the switching property [34]

   Biologically these functions signify the fact that the predatory rate decreases when the population of one prey species becomes rare compared to the population of the other prey species. Since logically the amount of food taken by a predator from one of its prey effected by the amount of food taken from the other prey species, consequently we will use the following modified switching functions, which are give by $\frac{P_1 SY}{1+\left(\frac{I}{S}\right)^2+\left(\frac{S}{I}\right)^2}$ and $\frac{P_2 IY}{1+\left(\frac{I}{S}\right)^2+\left(\frac{S}{I}\right)^2}$ instead of the above functions.

According to the above hypothesis the dynamics of a prey-predator model involving an *SIS* epidemic disease in prey population can be describe by the following set of nonlinear differential equations:

$$\frac{dS}{dt} = rS\left(1-\frac{S+I}{K}\right)-(\beta I+C)S+\gamma I-\frac{P_1 SY}{1+\left(\frac{I}{S}\right)^2+\left(\frac{S}{I}\right)^2}$$

$$\frac{dI}{dt} = (\beta I+C)S-d_1 I-\gamma I-\frac{P_2 IY}{1+\left(\frac{I}{S}\right)^2+\left(\frac{S}{I}\right)^2} \qquad (.1)$$

$$\frac{dY}{dt} = -d_2 Y+\frac{e_1 P_1 SY+e_2 P_2 IY}{1+\left(\frac{I}{S}\right)^2+\left(\frac{S}{I}\right)^2}$$

here $S(0)>0$, $I(0)>0$ and $Y(0)>0$. Obviously the interaction functions of the system (1) are continuous and have continuous partial derivatives on the region. $R_+^3 = \{(S,I,Y)\in R^3 : S(0)>0, I(0)>0, Y(0)>0\}$. Therefore these functions are Lipschitzian on R_+^3, and hence the solution of the system (1) exists and is unique. Further, in the following theorem the boundedness of the solution of the system (1) in R_+^3 is established.

Theorem (1): All the solutions of the system (1) are uniformly bounded.

Proof: Let $W = S + I + Y$ then we get

$$\frac{dW}{dt} = \frac{dS}{dt}+\frac{dI}{dt}+\frac{dY}{dt}$$

So, by substituting the values of $\frac{dS}{dt}$, $\frac{dI}{dt}$, $\frac{dY}{dt}$ and then simplifying the resulting terms we get

$$\frac{dW}{dt} \le S(r+1)-S-d_1 I-d_2 Y$$

Now since S growth logistically in the absence of other species with internist growth rate r and carrying capacity K then it is easy to verify that S is bounded above by $\hat{K} = \max\{S(0), K\}$. Further let $\hat{d} = \min\{1, d_1, d_2\}$ then we get:

$$\frac{dW}{dt} \leq \hat{K}(r+1) - \hat{d}W$$

thus as $t \to \infty$ it is obtain that $0 \leq W \leq \frac{\hat{K}(r+1)}{\hat{d}}$. Hence all solutions of system (1) are uniformly bounded and therefore we have finished the proof. ■

The equilibrium points:

In this section all possible equilibrium points of system (1) are presented:

1. Although, system (1) is not well defined at the origin and the vanishing equilibrium point $E_0(0,0,0)$ is not exist mathematically, the extinction of all species in any environment is still possible and hence the equilibrium point E_0 exists biologically and need to study.

2. The predator free equilibrium point that denoted by $E_1 = (\hat{S}, \hat{I}, 0)$, where

$$\hat{S} = \frac{-B_1 + \sqrt{B_1^2 - 4B_2}}{2} \text{ and } \hat{I} = \frac{C\hat{S}}{d_1 + \gamma - \beta\hat{S}} \tag{2a}$$

here $B_1 = -\frac{Cr + r(\gamma + d_1 + K\beta)}{r\beta}$ and $B_2 = \frac{rK(d_1+\gamma) - Cd_1K}{r\beta}$, exists uniquely in the $Int.R_+^2$ of $SI-$plane provided that the following condition holds

$$\beta\hat{S} < d_1 + \gamma < \frac{Cd_1}{r} \tag{2b}$$

3. The positive equilibrium point $E_2 = (S^*, I^*, Y^*)$ exists uniquely in $Int.R_+^3$ provided that the follow algebraic system has a unique positive solution given by S^*, I^* and Y^*.

$$rS\left(1 - \frac{S+I}{K}\right) - (\beta I + C)S + \gamma I - \frac{P_1 S^3 I^2 Y}{S^2 I^2 + I^4 + S^4} = 0$$

$$(\beta I + C)S - d_1 I - \gamma I - \frac{P_2 I^3 S^2 Y}{S^2 I^2 + I^4 + S^4} = 0$$

$$-d_2 Y + \frac{e_1 P_1 S^3 I^2 Y + e_2 P_2 I^3 S^2 Y}{S^2 I^2 + I^4 + S^4} = 0$$

3.4 The stability analysis of system (1):

In this section the stability of each equilibrium point of system (1) is studied and the results are summarized as follows:

Since the Jacobian matrix of system (1) near the trivial equilibrium $E_0 = (0,0,0)$ is not defined. Hence the dynamical behavior of system (1) around E_0 is studied by using the technique of Venturino (2008) and Arino [1]. Now, rewrite system (1) in R^N as follows:

$$\frac{dX}{dt} = H(X(t)) + Q(X(t)) \tag{3}$$

in which H is C^1 outside the origin and homogenous of degree 1, thus

$$H(sX) = sH(X)$$

For all $s \geq 0, X \in R^N$, and Q is a C^1 function such that in the vicinity of the origin we have $Q(X) = o(X)$.

To study the behavior of the system at the origin point, we use $\|\cdot\|$ that denotes the Euclidian norm on R^N, and $\langle \cdot,\cdot \rangle$ denotes the associated inner product, in the case of our model, $N = 3$. Let

$$X = (x_1, x_2, x_3) = (S, I, Y);\ H(X) = (H_1(X), H_2(X), H_3(X));$$
$$Q(X) = (Q_1(X), Q_2(X), Q_3(X)).$$

Therefore, the functions H_i and $Q_i\ (i = 1,2,3)$ are given by

$$H_1(X) = rx_1 - Cx_1 + \gamma x_2;\ H_2(X) = Cx_1 - d_1x_2 - \gamma x_2;\ H_3(X) = -d_2x_3$$

$$Q_1(X) = \frac{-rx_1^2 + rx_1x_2}{K} - \beta x_2x_1 - \frac{P_1x_1^3x_2^2x_3}{x_1^2x_2^2 + x_2^4 + x_1^4};$$

$$Q_2(X) = \beta x_2x_1 - \frac{P_2x_2^3x_1^2x_3}{x_1^2x_2^2 + x_2^4 + x_1^4};$$

$$Q_3(X) = \frac{e_1P_1x_1^3x_2^2x_3 + e_2P_2x_2^3x_1^2x_3}{x_1^2x_2^2 + x_2^4 + x_1^4}$$

Let $X(t)$ be a solution of system (3). Assume that $\lim_{t\to\infty} \inf \|X(t)\| = 0$ and X is bounded. Then it is possible to extract the sequences $X(t_n + \cdot), t_n \to \infty$, from the family $(X(t+\cdot))_{t\geq 0}$ such that $X(t_n + .) \to 0$ locally uniformly on $s \in R$.

Define $y_n(s) = \dfrac{X(t_n + s)}{\|X(t_n + s)\|}$ (4)

Recall that, $Q(X) = o(X)$ in the vicinity of the origin. Then Q can be written as

$$Q(X) = \|X\|^2 O(1)\ . \qquad (5)$$

We have

$$\frac{dX(t_n + s)}{ds} = H(X(t_n + s)) + Q(X(t_n + s)). \qquad (6)$$

from (4) we have

$$X(t_n + s) = y_n(s)\|X(t_n + s)\| = y_n(s) \cdot \langle X(t_n + s), X(t_n + s)\rangle^{\frac{1}{2}} \qquad (7)$$

On the other hand we have

$$\frac{d}{ds}\langle X(t_n + s), X(t_n + s)\rangle = 2\left\langle X(t_n + s), \frac{dX(t_n + s)}{ds}\right\rangle \qquad (8)$$

So, by take the derivative of $X(t_n + s)$ in Eq. (7) with respect to *s*, and using Eq. (8) we obtain

$$\frac{dX(t_n + s)}{ds} = \frac{dy_n(s)}{ds}\|X(t_n + s)\| + \frac{y_n(s)}{\|X(t_n + s)\|}\left\langle X(t_n + s), \frac{dX(t_n + s)}{ds}\right\rangle$$

Therefore we have

$$H(X(t_n+s))+Q(X(t_n+s))=\frac{dy_n(s)}{ds}\|X(t_n+s)\|$$
$$+\frac{y_n(s)}{\|X(t_n+s)\|}\langle X(t_n+s),H(X(t_n+s))+Q(X(t_n+s))\rangle$$

Now dividing by $\|X(t_n+s)\|$ and replacing $\frac{X(t_n+s)}{\|X(t_n+s)\|}$ by $y_n(s)$, then simplifying the resulting terms give

$$\frac{dy_n(s)}{ds}=H(y_n(s))-y_n(s)\langle y_n(s),H(y_n(s))\rangle+\|X(t_n+s)\|$$
$$\times\left\{\frac{Q(X(t_n+s))}{\|X(t_n+s)\|^2}-y_n(s)\left\langle y_n(s),\frac{Q(X(t_n+s))}{\|X(t_n+s)\|^2}\right\rangle\right\}$$

Which is equivalent to

$$\frac{dy_n}{ds}=\left[H(y_n(s))-y_n(s)\langle y_n(s),H(y_n(s))\rangle\right]$$
$$+\|X(t_n+s)\|\left[Q(y_n(s))-y_n(s)\langle y_n(s),Q(y_n(s))\rangle\right]$$

Clearly, y_n is bounded, $\|y_n(s)\|=1,\forall s,$ and $\frac{dy_n}{ds}$ is bounded too. So, applying the Ascoli – Arzela theorem [2], one can extract from $y_n(s)$ a subsequence-also denoted by $y_n(s)$, which converges locally uniformly on R towards some function y, such that:

$$\|X(t_n+s)\|\left[Q(y_n(s))-y_n(s)\langle y_n(s),Q(y_n(s))\rangle\right]\xrightarrow[t_n\to\infty]{}0$$

and y satisfies the following system:

$$\frac{dy}{dt}=H(y(t))-y(t)\langle y(t),H(y(t))\rangle\,;\ \|y(t)\|=1,\ \forall t \qquad (9)$$

Equation (9) is defined for all $t\in R$.

Let us, for a moment, focus on the study of equation (9). The steady states of H are vectors V satisfying

$$H(V)=V\langle V,H(V)\rangle$$

This is a so-called nonlinear eigenvalue. Note that the equation can be alternatively written as

$$H(V)=\mu V \qquad (10)$$

with $\|V\|=1$; it then holds that $\mu=\langle V,H(V)\rangle$. These stationary solutions correspond to fixed directions that the trajectories of equation (9) may reach asymptotically. Further more equation (10) can be written as

$$\begin{aligned}&(\mu-r+C)v_1-\gamma v_2=0\\&Cv_1-(\mu+d_1+\gamma)v_2=0\\&(\mu+d_2)v_3=0\end{aligned} \qquad (11)$$

Now, we are in a position to discuss in detail the possibility of reaching the origin following fixed directions.

Case (1) $v_3=0$

(a) $v_1=0$ and $v_2\neq 0$: in this case, there is a possibility to reach the origin following the $I-$axis when $\mu=-d_1$ with $\gamma=0$.

(b) $v_1 \neq 0$ and $v_2 = 0$: in this case there is a possibility to reach the origin following $S-$ axis when $\mu = r$ with $C = 0$.

(c) $v_1 \neq 0$ and $v_2 \neq 0$: in this case, we obtain different results depending on the parameters:

Sub case (1): there is a possibility to reach the origin if

$$(d_1 + \gamma + C - r)^2 + 4r(d_1 + \gamma) > 4Cd_1$$

$$\mu = -\frac{(d_1+\gamma+C-r)}{2} + \frac{1}{2}\left[(d_1 + \gamma + C - r)^2 + (r(d_1 + \gamma) - Cd_1)\right]^{\frac{1}{2}}$$

Sub case (2): we can not reach the origin otherwise.

Case (2) $v_3 \neq 0$

In this case, we obtain different results depending on the parameters as these given case 1 above with $\mu = -d_2$.

The Jacobian matrix of system (1) at the predator free equilibrium point E_1 can be written as:

$$J(E_1) = \begin{pmatrix} r\left(1 - \frac{2\hat{S}+\hat{I}}{K}\right) - \left(\beta\hat{I} + C\right) & \frac{-r\hat{S}}{K} - \beta\hat{S} + \gamma & \dfrac{-P_1\hat{S}}{1 + \left(\frac{\hat{I}}{\hat{S}}\right)^2 + \left(\frac{\hat{S}}{\hat{I}}\right)^2} \\ \beta\hat{I} + C & \beta\hat{S} - d_1 - \gamma & \dfrac{-P_2\hat{I}}{1 + \left(\frac{\hat{I}}{\hat{S}}\right)^2 + \left(\frac{\hat{S}}{\hat{I}}\right)^2} \\ 0 & 0 & -d_2 + \dfrac{e_1P_1\hat{S} + e_2P_2\hat{I}}{1 + \left(\frac{\hat{I}}{\hat{S}}\right)^2 + \left(\frac{\hat{S}}{\hat{I}}\right)^2} \end{pmatrix}$$

Consequently, the characteristic equation can be written as

$$\left(\lambda_1{}^2 - T\lambda_1 + D\right)\left[-d_2 + \frac{e_1p_1\hat{S} + e_2p_2\hat{I}}{1 + \left(\frac{\hat{I}}{\hat{S}}\right)^2 + \left(\frac{\hat{S}}{\hat{I}}\right)^2} - \lambda_1\right] = 0$$

here

$$T = r\left(1 - \frac{2\hat{S}+\hat{I}}{K}\right) - \left(\beta\hat{I} + C\right) + \left(\beta\hat{S} - d_1 - \gamma\right)$$

$$D = \left[r\left(1 - \frac{2\hat{S}+\hat{I}}{K}\right) - (\beta\hat{I} + C)\right]\left(\beta\hat{S} - d_1 - \gamma\right) - \left(\beta\hat{I} + C\right)\left(\frac{-r\hat{S}}{K} - \beta\hat{S} + \gamma\right)$$

Clearly the eigenvalues of this Jacobian matrix satisfy the following relationships:

$$T = \lambda_{1S} + \lambda_{1I},\ D = \lambda_{1S}.\lambda_{1I},\ \text{and}\ \lambda_{1Y} = -d_2 + \frac{e_1p_1\hat{S} + e_2p_2\hat{I}}{1 + \left(\frac{\hat{I}}{\hat{S}}\right)^2 + \left(\frac{\hat{S}}{\hat{I}}\right)^2} \quad (12)$$

Accordingly the equilibrium point E_1 is locally asymptotically stable provided that:

$$r\left(1 - \frac{2\hat{S}+\hat{I}}{K}\right) < \left(\beta\hat{I} + C\right) \quad (13a)$$

$$\frac{K\gamma}{r+\beta K}<\hat{S} \qquad (13b)$$

$$\frac{e_1 p_1\hat{S}+e_2 p_2\hat{I}}{1+\left(\frac{\hat{I}}{\hat{S}}\right)^2+\left(\frac{\hat{S}}{\hat{I}}\right)^2}<d_2 \qquad (13c)$$

However, it is unstable point otherwise.

The Jacobian matrix of coexistence equilibrium point E_2 can be written as:

$$J(E_2)=(c_{ij})_{3\times 3}$$

where:

$$c_{11}=r\left(1-\frac{2S^*+I^*}{K}\right)-M_3-\frac{P_1 S^{*^2} I^{*^2} Y^*\left(S^{*^2} I^{*^2}-S^{*^4}+3I^{*^4}\right)}{M_4^2}$$

$$c_{12}=\frac{-rS^*}{K}-\beta S^*+\gamma-\frac{2P_1 S^{*^3} I^* Y^*\left(S^{*^4}-I^{*^4}\right)}{M_4^2}$$

$$c_{13}=\frac{-P_1 S^{*^3} I^{*^2}}{M_4}<0,\ c_{21}=M_3-\frac{2P_2 S^* I^{*^3} Y^*\left(I^{*^4}-S^{*^4}\right)}{M_4^2}$$

$$c_{22}=\beta S^*-d_1-\gamma-\frac{P_2 S^{*^2} I^{*^2} Y^*\left(S^{*^2} I^{*^2}-I^{*^4}+3S^{*^4}\right)}{M_4^2} \quad c_{23}=\frac{-P_2 S^{*^2} I^{*^3}}{M_4}<0$$

$$c_{31}=\frac{Y^* I^{*^2} S^*\left(M_4 M_5-2S^{*^2} M_6\left[I^{*^2}+2S^{*^2}\right]\right)}{M_4^2}$$

$$c_{32}=\frac{Y^* I^* S^{*^2}\left[M_4 M_7-2I^{*^2} M_6\left(S^{*^2}+2I^{*^2}\right)\right]}{M_4^2}$$

$$c_{33}=0.$$

where $M_3=\beta I^*+C$, $M_4=S^{*^2} I^{*^2}+I^{*^4}+S^{*^4}$.

$M_5=3e_1 P_1 S^*+2e_2 P_2 I^*$, $M_6=e_1 P_1 S^*+e_2 P_2 I^*$

and $M_7=2e_1 P_1 S^*+3e_2 P_2 I^*$.

Then the characteristic equation of $J(E_2)$ can be written as:

$$\lambda^3+C_1\lambda^2+C_2\lambda+C_3=0$$

here $C_1=-(c_{11}+c_{22})$, $C_2=c_{11}c_{22}-c_{12}c_{21}-c_{13}c_{31}-c_{23}c_{32}$

$$C_3=c_{31}(c_{13}c_{22}-c_{12}c_{23})+c_{32}(c_{11}c_{23}-c_{13}c_{21})$$

However

$$\Delta = C_1C_2 - C_3 = (c_{11} + c_{22})(c_{12}c_{21} - c_{11}c_{22}) + c_{31}(c_{11}c_{13} + c_{23}c_{12}) + c_{32}(c_{22}c_{23} + c_{13}c_{21})$$

According to Routh-Hurwitz criterion the equilibrium point E_2 is locally asymptotically stable provided that $C_1 > 0$,$C_3 > 0$ and $\Delta = C_1C_2 - C_3 > 0$. Hence straightforward computation show that the positive equilibrium point E_2 is locally asymptotically stable provided that

$$K < 2S^* + I^* \tag{14a}$$

$$I^{*^4} < S^{*^4} < 3I^{*^4} \tag{14b}$$

$$\frac{K\gamma}{r+K\beta} < S^* < \frac{d_1+\gamma}{\beta} \tag{14c}$$

$$M_4 > \max.\left\{\frac{2S^{*^2}M_6\left(I^{*^2}+2S^{*^2}\right)}{M_5}, \frac{2I^{*^2}M_6\left(S^{*^2}+2I^{*^2}\right)}{M_7}\right\} \tag{14d}$$

$$\left(P_1S^*\left(\beta S^* - d_1 - \gamma\right) - P_2I\gamma\right)M_4^2 < 2P_2P_1S^{*^5}I^{*^2}Y^*\left(I^{*^2} + 3S^{*^2}\right) \tag{14e}$$

$$P_2I^*\left(\left(d_1 + \gamma - \beta S^*\right) - M_3P_1S^*\right)M_4^2 > - S^{*^2}I^{*^3}Y^*\left(P_2^2\left(S^{*^2}I^{*^2} - I^{*^4} + 3S^{*^4}\right) + 2P_1^2\left(I^{*^4} - S^{*^4}\right)\right) \tag{14f}$$

Clearly, the condition (14a)-(14c) guarantee that c_{11}, c_{12} and c_{22} are negative while c_{21} is positive. However, condition (14d) guarantee that c_{31} and c_{32} are positive and hence $C_1 > 0$. Further the conditions (14a)-(14d) with condition (14e) guarantee that $C_3 > 0$. Finally the condition (14a)-(14d) with condition (14e) guarantee that $C_1C_2 - C_3 > 0$.

Theorem (3): Assume that E_1 is a locally asymptotically stable point in R_+^3 then E_1 is globally asymptotically stable on the sub region of R_+^3 that satisfy the following conditions:

$$G_2^2 < 4G_1G_3 \tag{15a}$$

$$\left(\sqrt{G_1}U_1 - \sqrt{G_3}U_2\right)^2 > \frac{Y\left(SP_1N_1 + IP_2N_2\right)}{1+\left(\frac{I}{S}\right)^2 + \left(\frac{S}{I}\right)^2} \tag{15b}$$

here $G_1 = C - r + \left(\frac{r}{K} + \beta\right)\hat{I} + \frac{r}{K}\hat{S}$, $G_2 = \gamma + C - \left(\frac{r}{K} + \beta\right)S + \beta\hat{I}$

$G_3 = d_1 + \gamma - \beta S$, $U_1 = S - \hat{S}$.

$U_2 = I - \hat{I}$, $N_1 = \hat{S} + e_1$, $N_2 = \hat{I} + e_2$

Proof: Consider the following positive definite function:

$$L_1 = \frac{\left(S-\hat{S}\right)^2}{2} + \frac{\left(I-\hat{I}\right)^2}{2} + Y$$

Clearly, $L_1 : R_+^3 \to R$ is continuously differentiable function so that $L_1\left(\hat{S}, \hat{I}, 0\right) = 0$ and $L_1(S, I, 0) > 0$ for all $(S, I, Y) \in R_+^3$ with $(S, I, Y) \neq \left(\hat{S}, \hat{I}, 0\right)$.

Therefore by differentiating this function with respect to the variable t we get:

$$\frac{dL_1}{dt} = \left(S - \hat{S}\right)\frac{dS}{dt} + \left(I - \hat{I}\right)\frac{dI}{dt} + \frac{dY}{dt}$$

Substituting the value of $\frac{dS}{dt}$, $\frac{dI}{dt}$ and $\frac{dY}{dt}$ in this equation and then simplifying the resulting terms we obtain:

$$\frac{dL_1}{dt} \leq -G_1 U_1{}^2 + G_2 U_1 U_2 - G_3 U_2{}^2 + \frac{Y(SP_1N_1 + IP_2N_2)}{1 + \left(\frac{I}{S}\right)^2 + \left(\frac{S}{I}\right)^2}$$

So, by using condition (15a) we obtain that:

$$\frac{dL_1}{dt} \leq -\left(\sqrt{G_1}U_1 - \sqrt{G_3}U_2\right)^2 + \frac{Y(SP_1N_1 + IP_2N_2)}{1 + \left(\frac{I}{S}\right)^2 + \left(\frac{S}{I}\right)^2}$$

Further according to condition (3.15b) it is easy to verify that $\frac{dL_1}{dt} < 0$, and hence L_1 is a Lyapnuov function. Thus E_1 is globally asymptotically stable on the sub region of R_+^3 that satisfy the given conditions. ■

Theorem (4): Assume that E_2 is a locally asymptotically stable point in R_+^3, then E_2 is globally asymptotically stable on the sub region of R_+^3 that satisfies the following condition:

$$G_5{}^2 < 4G_4G_6 \qquad (16a)$$

$$\left(\sqrt{G_4}U_3 - \sqrt{G_6}U_4\right)^2 > \frac{Y(SP_1N_3 + IP_2N_4)}{1 + \left(\frac{I}{S}\right)^2 + \left(\frac{S}{I}\right)^2} + \frac{Y^*\left(P_1S^*N_5 + P_2I^*N_6\right)}{1 + \left(\frac{I^*}{S^*}\right)^2 + \left(\frac{S^*}{I^*}\right)^2} \qquad (16b)$$

where $G_4 = C - r + \left(\frac{r}{K} + \beta\right)I^* + \frac{r}{K}S^*$, $G_5 = \left(\frac{r}{K} + \beta\right)S - \beta I^* - C - \gamma$,

$G_6 = d_1 + \gamma - \beta S$, $U_3 = S - S^*$.

$U_4 = I - I^*$, $N_3 = S^* + e_1Y^*$.

$N_4 = I^* + e_2Y^*$, $N_5 = S + e_1Y^*$, $N_6 = I + e_2Y^*$.

Proof: Consider the following positive definite function:

$$L_2 = \frac{\left(S - S^*\right)^2}{2} + \frac{\left(I - I^*\right)^2}{2} + \frac{\left(Y - Y^*\right)^2}{2}$$

Clearly, $L_2 : R_+^3 \to R$ is continuously differentiable function so that $L_2\left(S^*, I^*, Y^*\right) = 0$ and $L_2(S, I, Y) > 0$; $\forall (S, I, Y) \in R_+^3$ with $(S, I, Y) \neq \left(S^*, I^*, Y^*\right)$.

Therefore by differentiating this function with respect to the variable t we get:

$$\frac{dL_2}{dt} = \left(S - S^*\right)\frac{dS}{dt} + \left(I - I^*\right)\frac{dI}{dt} + \left(Y - Y^*\right)\frac{dY}{dt}$$

Substituting the value of $\frac{dS}{dt}$, $\frac{dI}{dt}$ and $\frac{dY}{dt}$ in this equation and then simplifying the resulting terms we obtain:

$$\frac{dL_2}{dt} \le -G_4U_3^2 + G_5U_3U_4 - G_6U_4^2 + \frac{Y\left(SP_1N_3 + IP_2N_4\right)}{1+\left(\frac{I}{S}\right)^2 + \left(\frac{S}{I}\right)^2} + \frac{Y^*\left(P_1S^*N_5 + P_2I^*N_6\right)}{1+\left(\frac{I^*}{S^*}\right)^2 + \left(\frac{S^*}{I^*}\right)^2}$$

So, by using condition (16a) we obtain that:

$$\frac{dL_2}{dt} \le -\left(\sqrt{G_4}U_3 - \sqrt{G_6}U_4\right)^2 + \frac{Y\left(SP_1N_3 + IP_2N_4\right)}{1+\left(\frac{I}{S}\right)^2 + \left(\frac{S}{I}\right)^2} + \frac{Y^*\left(P_1S^*N_5 + P_2I^*N_6\right)}{1+\left(\frac{I^*}{S^*}\right)^2 + \left(\frac{S^*}{I^*}\right)^2}$$

Now according to condition (16b) it is easy to verify that $\frac{dL_2}{dt} < 0$, and hence L_2 is a Lyapnuov function. Thus E_2 is globally asymptotically stable on the sub region of R_+^3 that satisfy the given conditions.

II. NUMERICAL SIMULATION

In this section the dynamical behavior of system (1) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: to investigate the effect of varying the value of each parameter (especially the switching rate) on the dynamical behavior of system (1) and toconfirm our obtained analytical results. Now for the following set of hypothetical parameters values:

$$\begin{aligned} &r = 1, K = 200, \beta = 0.2, C = 0.1, \gamma = 0.4, P_1 = 1, P_2 = 1, \\ &d_1 = 0.1, d_2 = 0.1, e_1 = 0.5, e_2 = 0.6 \end{aligned} \quad (17)$$

The trajectory of system (1) is drawn in the Fig.(1) for different initial point

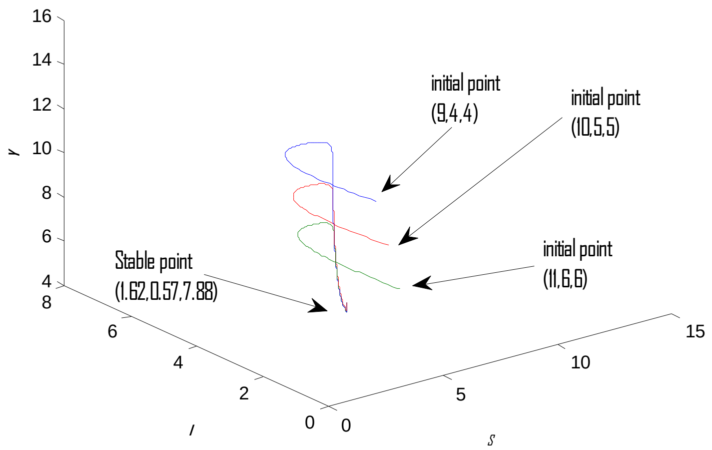

Fig.(1): Phase plot of system (1) starting from different initial points.

In the above figure, system (1) approaches asymptotically to the stable coexistence equilibrium point starting from different initial points, which indicates the global stability of the positive equilibrium point.

Note that in time series figures, we will use throughout this section that: blue color for describing the trajectory of S, green color for describing the trajectory of I and red color for describing the trajectory of Y.

Now in order to discuss the effect of varying the intrinsic growth rate r on the dynamical behavior of system (1), the system is solved numerically for different values of parameter $r = 0.1, 2$ keeping other parameters fixed as given in Eq.(17) and then the solution of system (1) is drawn in Fig.(2a)-(2b) while their time series are drawn in Fig. (3a)-(3b) respectively.

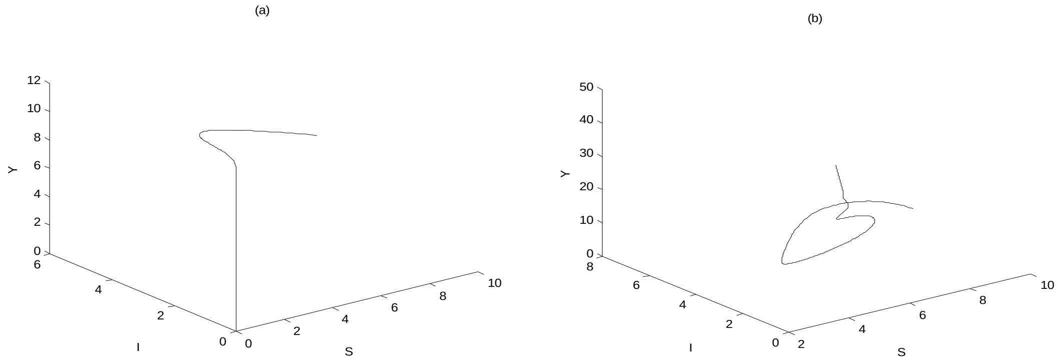

Fig.(2): Phase plots of system (1) for the data given by Eq.(17) (a) System (1) approaches asymptotically to E_0 when $r = 0.1$. (b) System (1) approaches asymptotically to coexistence equilibrium point when $r=2$

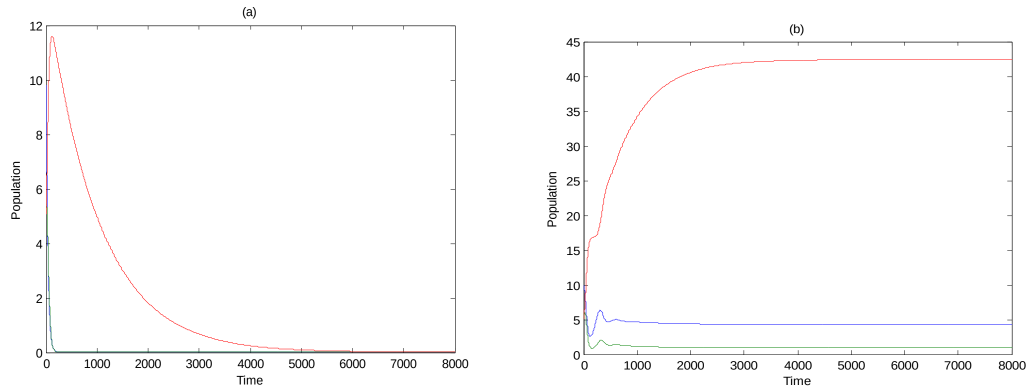

Fig. (3): Time series for the solution of system (3.1). (*a*) Time series for the attractor in Fig.(2*a*) (*b*) Time series for the attractor in Fig.(2b).

Clearly from the Figs. (2*a*) and (3a), the solution of system (1) approaches asymptotically to the vanishing equilibrium point E_0, which confirm our analytical results regarding to possibility of approaching of the solution to the vanishing equilibrium point.

The effect of varying the switching rate P_2 of infected prey species on the dynamics of system (1) is studied and the trajectories of system (1) are drawn in Fig. (4a)-(4b) for the values $P_2 = 0.6, 0.36$ respectively, while their time series are drawn in Fig. (5a)-(5b) respectively.

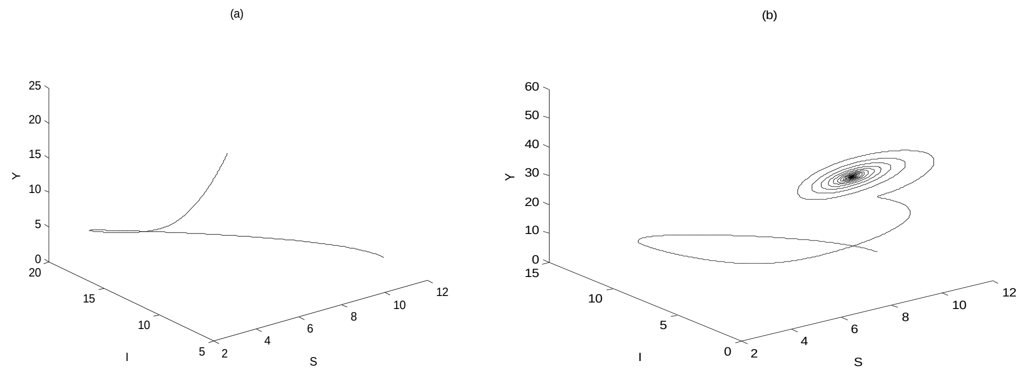

Fig.(4): Phase plots of system (1) for the data given in Eq.(17). (a) System (1) approaches asymptotically to coexistence equilibrium point for $P_2 = 0.6$. (b) System (1) approaches to periodic attractor for $P_2 = 0.36$.

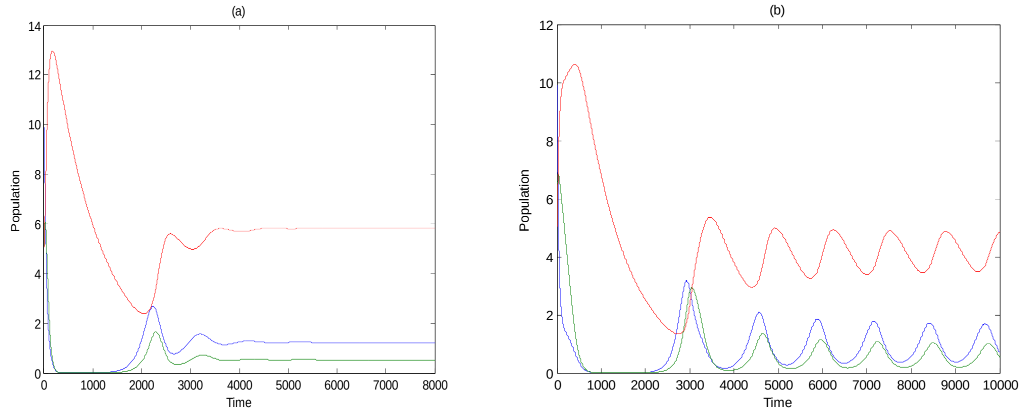

Fig.(5): Time series for the solution of system (1). (a) Time series for the attractor in Fig. (4a) (b) Time series for the attractor in Fig.(4b).

According to the above figures the system loses it stability of positive equilibrium point and the solution approaches to periodic dynamics in $Int.R_+^3$ due to decreasing in the switching rate constant P_2.

The effect of varying death rate of predator d_2 on the dynamical behavior of system (1) is studied and the trajectories of system (1) are drawn in Fig. (6a)-(6b) for the values $d_2 = 0.8, 0.4$ respectively, while their time series are drawn in Fig.(7a) -(7b) respectively.

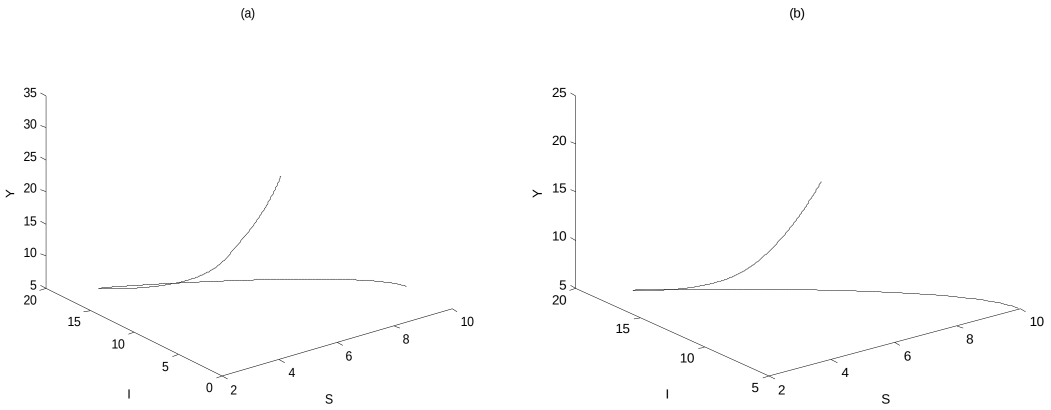

Fig. (6): Phase plots of system (1) for the data given in Eq. (17) (*a*) System (1) approaches asymptotically to predator free equilibrium point in the *SI*-plane for $d_2 = 0.8$. (b) System (1) approaches asymptotically to coexistence equilibrium point for $d_2 = 0.4$.

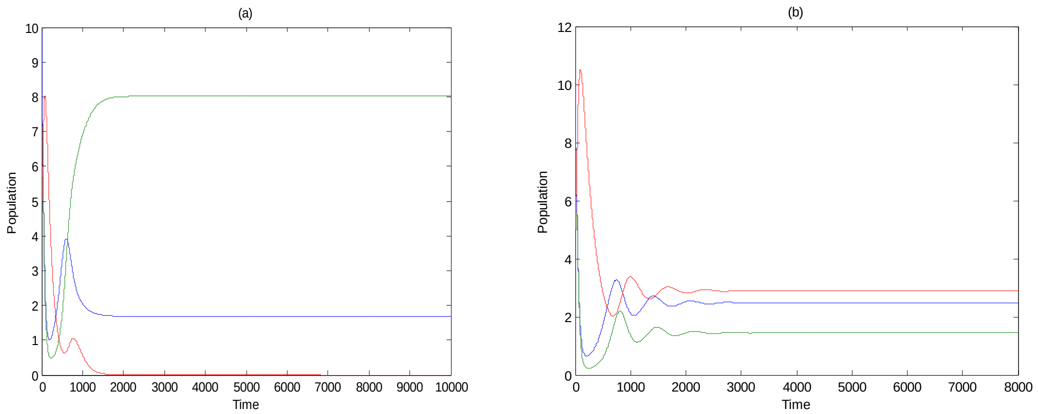

Fig.(7): Time series for the solution of system (1). (*a*) Time series for the attractor in Fig.(6a) (*b*) Time series for the attractor in Fig.(6b) .

According to these figures, increasing the predator death rate causes extinction in predator species of system (1).

2.5 Discussion and conclusion:

In this chapter, we proposed and analyzed an eco-epidemiological model that described the dynamical behavior of a prey-predator model with modify switching function. The model include three non-linear differential equations that describe the dynamics of three different populations namely predator *Y* susceptible prey *S* infected prey *I*. The boundedness of the system (1) has been discussed. The conditions for existence and stability of each equilibrium points are obtained. To understand the effect of varying each parameter, system (1) is solved numerically and then the obtained results can be summarized as follow:

1. Decreasing the intrinsic growth rate r causes extinction in all species and the solution of the system (1) approaches asymptotically to the vanishing equilibrium point E_0 for $r \leq 0.2$. However increasing the intrinsic growth rate causes persists of all species and the solution of system (1) approaches asymptotically to the positive equilibrium point E_2.

2. Decreasing the values of predator switching rate P_2 in the range $P_2 \leq 0.39$ causes destabilizing of the coexistence equilibrium point E_2 in the $Int.R_+^3$ and the solution of system (1) approaches asymptotically to periodic dynamics in $Int.R_+^3$.

3. Increasing the values of death rate of predator d_2 causes extinction in predator species and the solution of the system (1) approaches asymptotically to the predator free equilibrium point E_1 for $d_2 \geq 0.73$

4. Finally it is observed that, varying each of the values of parameters $K, \beta, P_1, \gamma, e_1, C, e_2, d_1$ has no effect on the dynamics of the system (1) and the system still approaches asymptotically to coexistence equilibrium point.

REFERENCES

[1] Arino O., Abdllaoui A., Mikram J., and Chattopadhyay J., Infection in prey population may act as a biological control in ratio-dependent predator-prey models, Nonlinearity, 17, 1101-1116, (2004).

[2] Friedman A., Foundations of Modern Analysis, New York, Dover Publications, Inc., (1982).

[3] Khan, Q.A.J., Balakrishnan, E., Wake, G.C., Analysis of a predator-prey system with predator switching. Bull. Math. Biol. 66,109-123,(2004).

[4] Malchow, H., Hilker, F.M., Sarkar, R.R., Brauer, K., Spatiotemporal patterns in an excitable plankton system with lysogenic viral infections. Math. Comput. Model. 42, 1035-1048,(2005).

[5] Mukhopadhyay. B. , Bhattacharyya .R. Role of Predator switching in an eco-epidemiological model with disease in the prey . Ecological Modelling , 931-939,(2009).

[6] Tansky, M., Switching effects in prey-predator system. J. Theor. Biol. 70, 263-271, (1978).