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# A Note on Riemann Zeta Function

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***Abstract***: The Riemann Zeta Function is an essential special function in analytical number theory; its connection to the distribution of prime numbers was the motivation to expand its domain from real numbers to complex numbers[3,4]. The importance of the Riemann Zeta Function is not only due to its crucial applications but furthermore due to the Riemann Hypothesis that remains unsolved till now. In this article basic properties are discussed and some important functional relationships are investigated.

***Keywords***: *Riemann Zeta Function, Gamma Function, Psi Function, Functional Equations, Xi Function.*

## I. INTRODUCTION

The Riemann Zeta Function is defined by $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ where $s=x+iy,$ this notation for complex number $s$ is due to Riemann and now it is a standard notation in this context [1], [2]. The series above is absolutely convergent for $R(s)=x>1,$ since

$$|\zeta(s)|=\left|\sum_{n=1}^{\infty}\frac{1}{n^s}\right|\le\sum_{n=1}^{\infty}\left|\frac{1}{n^s}\right|=\sum_{n=1}^{\infty}\frac{1}{|n^s|}=\sum_{n=1}^{\infty}\frac{1}{|n^x||n^{iy}|}=\sum_{n=1}^{\infty}\frac{1}{n^x|e^{iylogn}|}=\sum_{n=1}^{\infty}\frac{1}{n^x}$$

by Cauchy-Integral Theorem:

$$\sum_{n=1}^{\infty}\frac{1}{n^x}\leftrightarrow\int_1^{\infty}\frac{1}{t^x}dt=\frac{1}{1-x}t^{1-x}|_1^{\infty},\qquad for\ x=Re(s)>1.$$

## II. EULER PRODUCT REPRESENTATION

In this section we derive and prove the connection between prime numbers and the Riemann Zeta Function. Let $P$ denote the set of all prime numbers, then the following holds true.

***Theorem 2.1***

$$|\zeta(s)|=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p\in P}\frac{1}{1-p^{-s}}\qquad(1)$$

***Proof*** let $p\in P,$ i.e. $p$ is a prime number

*Method I:*

$$\prod_{p\in P}\frac{1}{1-p^{-s}}=\prod_{i=1}\sum_{k_i=0}^{\infty}\left(\frac{1}{p_i{}^s}\right)^{k_i}=\prod_{i=1}\sum_{k_i=0}^{\infty}\frac{1}{p_i{}^{sk_i}}=\sum_{k_1=0}^{\infty}\frac{1}{p_1{}^{sk_1}}\cdot\sum_{k_2=0}^{\infty}\frac{1}{p_2{}^{sk_2}}\cdots\sum_{k_N=0}^{\infty}\frac{1}{p_N{}^{sk_N}}\ \cdots$$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\left(\frac{1}{p_1{}^{k_1}}\frac{1}{p_2{}^{k_2}}\cdots\frac{1}{p_N{}^{k_N}}\cdots\right)^S$$

by Fundamental Theorem of Arithmetic

$$n=\prod p_1{}^{k_1}p_2{}^{k_2}\dots p_{N(n)}{}^{k_{N(n)}}$$

this implies

$$\prod_{p\in P}\frac{1}{1-p^{-s}}=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

□

*Method II* we multiply $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ by $\frac{1}{2^s}$ this implies

$$\zeta(s)\frac{1}{2^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}\frac{1}{2^s}=\sum_{n=1}^{\infty}\frac{1}{(2n)^s}$$

if we subtract from $\zeta(s)$ this yields

$$\zeta(s)-\zeta(s)\frac{1}{2^s}=\sum_{n=1}^{\infty}\frac{1}{n^s}-\sum_{n=1}^{\infty}\frac{1}{(2n)^s}$$

$$\zeta(s)\left(1-\frac{1}{2^s}\right)=\sum_{\substack{n=1\\ n\neq 2k}}^{\infty}\frac{1}{n^s}$$

we multiply the above expression by $\frac{1}{3^s}$ and get

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\frac{1}{3^s}=\sum_{\substack{n=1\\ n\neq 2k}}^{\infty}\frac{1}{n^s}\frac{1}{3^s}$$

then we subtract again to derive

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)=\sum_{\substack{n=1\\ n\neq 2k}}^{\infty}\frac{1}{n^s}-\sum_{\substack{n=1\\ n\neq 2k}}^{\infty}\frac{1}{(3n)^s}$$

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\ =\sum_{\substack{n=1\\ n\neq 2\\ n\neq 3}}^{\infty}\frac{1}{n^s}$$

by repeating the above steps with all prime numbers to conclude that

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\dots=\sum_{\substack{n=1\\ n\neq 2\\ n\neq 3\\ n\neq 5\\ \dots}}^{\infty}\frac{1}{n^s}=1$$

therefore

$$\zeta(s)=\frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\dots}=\prod_{p\in P}\frac{1}{1-p^{-s}}$$

□

**Corollary 2.1**: "Infinitude of Prime Numbers" if we let $s = 1$ in assertion (1) the following holds true

$$\prod_{p\in P}\frac{1}{1-p^{-1}}=\zeta(1)=\infty\,.$$

Next, we prove that the Prime Zeta Function $\sum_{p\in P}\frac{1}{p}$ is a divergent function.

***Theorem 2.2***

$$\sum_{p\in P}\frac{1}{p}\to\infty \qquad (2)$$

***Proof***. Since $\zeta(s)=\prod_{p\in P}\frac{1}{1-p^{-s}}$ then
$log\zeta(s)=log\prod_{p\in P}\frac{1}{1-p^{-s}}=\sum_{p\in P}log\frac{1}{1-p^{-s}}$, we know that:

$$log\frac{1}{1-x}=-\log(1-x)=-\left(-\sum_{n=1}^{\infty}\frac{x^n}{n}\right)=\sum_{n=1}^{\infty}\frac{x^n}{n}$$

replace $x$ in the expression above by $p^{-s}$ to provide

$$log\zeta(s)=\sum_{p\in P}\sum_{n=1}^{\infty}\frac{(p^{-s})^n}{n}=\sum_{p\in P}\sum_{n=1}^{\infty}\frac{p^{-sn}}{n}=\sum_{p\in P}p^{-s}+\sum_{p\in P}\sum_{n=2}^{\infty}\frac{p^{-sn}}{n}$$

$$=\sum_{p\in P}\frac{1}{p^s}+\sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{sn}} \qquad (3)$$

Now, the last term in the previous expression, satisfies the following

$$\left|\sum_{p\in P}\frac{1}{p^{sn}}\right|\le\sum_{p\in P}\frac{1}{|p^{sn}|}=\sum_{p\in P}\frac{1}{|p^{xn+iyn}|}=\sum_{p\in P}\frac{1}{|p^{xn}||p^{iyn}|}=\sum_{p\in P}\frac{1}{p^{xn}|e^{iynlogp}|}=\sum_{p\in P}\frac{1}{p^{xn}}$$

$$\left|\sum_{p\in P}\frac{1}{p^{sn}}\right|\le\sum_{p\in P}\frac{1}{p^{xn}}\le\sum_{p\in P}\frac{1}{p^{n}}\le\sum_{k=2}^{\infty}\frac{1}{k^n}<\sum_{k=2}^{\infty}\int_{k-1}^{k}\frac{1}{t^n}dt=\int_{1}^{\infty}\frac{1}{t^n}dt=\frac{1}{n-1}$$

therefore

$$\left|\sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{sn}}\right| \le \sum_{n=2}^{\infty}\frac{1}{n}\left|\sum_{p\in P}\frac{1}{p^{sn}}\right| \le \sum_{n=2}^{\infty}\frac{1}{n} \qquad (4)$$

by using the assertion (4), and letting $s \to 1$ (3) becomes

$$log\zeta(s) = \sum_{p\in P}\frac{1}{p^s} + \sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{sn}}$$

the left hand side goes to infinity and the last right hand side is bounded by 1, that implies:

$$\sum_{p\in P}\frac{1}{p} \to \infty$$

□

## III. THE PRIME COUNTING FUNCTION $\pi(x)$

***Theorem 3.1***

$$log\zeta(s) = \int_2^{\infty}\frac{s\pi(x)}{x(x^s-1)}dx \qquad (5)$$

***Proof.*** From (1) we can write:

$$log\zeta(s) = \sum_{n=2}^{\infty}[\pi(n)-\pi(n-1)]\log\frac{1}{1-n^{-s}}$$

$$= \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty}\pi(n-1)\log\frac{1}{1-n^{-s}}$$

$$= \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-(n+1)^{-s}}$$

$$= \sum_{n=2}^{\infty}\pi(n)[\log(1-(n+1)^{-s}) - \log(1-n^{-s})]$$

now,

$$\frac{d}{dx}\log(1-x^{-s}) = \frac{1}{1-x^{-s}}(sx^{-s-1}) = \frac{s}{x(x^s-1)}$$

this implies

$$\log(1-x^{-s}) = \int\frac{s}{x(x^s-1)}dx + c$$

$$\log\zeta(s) = \sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}dx = \sum_{n=2}^{\infty}\int_n^{n+1}\frac{s\pi(x)}{x(x^s-1)}dx = \int_2^{\infty}\frac{s\pi(x)}{x(x^s-1)}dx$$

□

## IV. SPECIAL VALUES OF ZETA

***Theorem 4.1*** "Basel Problem"

$$\zeta(2) = \frac{\pi^2}{6}$$

***Proof***. By the means of Taylor expansion of the sin function:

$$\sin s = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} \pm \cdots$$

let $s \to \pi s$:

$$\frac{\sin\ \pi s}{\pi s} = 1 - \frac{(\pi s)^2}{3!} + \frac{(\pi s)^4}{5!} - \frac{(\pi s)^6}{7!} \pm \cdots$$

$$\frac{\sin\ \pi s}{\pi s} = 1 - \frac{\pi^2}{3!}s^2 + \frac{\pi^4}{5!}s^4 - \frac{\pi^6}{7!}s^6 \pm \cdots$$

using the fact that:

$$\frac{\sin\ \pi s}{\pi s} = \left(1-\frac{s^2}{1^2}\right)\left(1-\frac{s^2}{2^2}\right)\left(1-\frac{s^2}{3^2}\right)\ldots$$

$$\frac{\sin\ \pi s}{\pi s} = 1 - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)s^2 + \left(\frac{1}{1^2 2^2} + \frac{1}{1^2 3^2} + \frac{1}{2^2 3^2} + \cdots\right)s^4 \left(\frac{1}{1^2 2^2 3^2} + \cdots\right)s^6 + \cdots$$

by equating coefficients the desired result follows □

***Theorem 4.2*** "Zeta of $2n$"

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \zeta(2n) s^{2n}$$

***Proof.*** we start by

$$\frac{\sin\ \pi s}{\pi s} = \prod_{k=1}^{\infty}\left(1 - \frac{s^2}{k^2}\right)$$

taking the logarithm of both sides, we have

$$\log \frac{\sin\ \pi s}{\pi s} = \log \prod_{k=1}^{\infty}\left(1 - \frac{s^2}{k^2}\right)$$

$$\log \sin \pi s = \log \pi s + \sum_{k=1}^{\infty} \log\left(1 - \frac{s^2}{k^2}\right)$$

we derive with respect to $s$

$$\frac{d}{ds}\log \sin \pi s = \frac{d}{ds}\left[\log \pi s + \sum_{k=1}^{\infty} \log\left(1 - \frac{s^2}{k^2}\right)\right]$$

$$\pi\ \frac{\cos \pi s}{\sin \pi s} = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{1}{\left(1 - \frac{s^2}{k^2}\right)}\left(-\frac{2s}{k^2}\right)$$

$$\pi s \cot \pi s = 1 + \sum_{k=1}^{\infty} \frac{1}{\left(1 - \frac{s^2}{k^2}\right)}\left(-\frac{2s^2}{k^2}\right)$$

$$\pi s \cot \pi s = 1 + 2s^2 \sum_{k=1}^{\infty} \frac{1}{(s^2 - k^2)}$$

$$\pi \cot \pi s = \frac{1}{s} + \sum_{k=1}^{\infty}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \sum_{k=-\infty}^{\infty} \frac{1}{s+k}$$

$$\pi s \cot \pi s = 1 + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{s^2}{k^2}\right)^n\left(-\frac{2s^2}{k^2}\right)$$

$$\pi s \cot \pi s = 1 - 2\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left(\frac{s^2}{k^2}\right)^{n+1}$$

$$\pi s \cot \pi s = 1 - 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^{2n}} s^{2n} = 1 - 2\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}\frac{1}{k^{2n}}\right)s^{2n}$$

that is:

$$\pi s \cot \pi s = 1 - 2\sum_{n=1}^{\infty} \zeta(2n) s^{2n}$$

which completes the proof □

***Theorem 4.3*** " the connection to Bernoulli's numbers"

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}\beta_{2n}}{2.(2n)!}$$

***Proof.***

$$\pi s \cot \pi s = \pi s\ \frac{\cos \pi s}{\sin \pi s} = \pi s\ \frac{e^{i\pi s} + e^{-i\pi s}}{2}\ \frac{2i}{e^{i\pi s} - e^{-i\pi s}}$$

$$= \pi s i\ \frac{e^{i\pi s} + e^{-i\pi s}}{e^{i\pi s} - e^{-i\pi s}} = \pi s i\ \frac{e^{2i\pi s} + 1}{e^{2i\pi s} - 1}$$

$$= \pi si \, \frac{e^{2i\pi s} - 1 + 2}{e^{2i\pi s} - 1} = i\pi s\left(1 + \frac{2}{e^{2i\pi s} - 1}\right)$$

this yield

$$\pi s \cot \pi s = i\pi s + \frac{2i\pi s}{e^{2i\pi s} - 1} \qquad (6)$$

we know that:

$$\frac{s}{e^s - 1} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} s^n \quad \rightarrow \quad \frac{s}{\sum_{n=1}^{\infty} \frac{1}{n!} s^n} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} \qquad (7)$$

that is equivalent to

$$s = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} s^n . \sum_{n=1}^{\infty} \frac{1}{n!} s^n$$

dived by $s$

$$1 = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} s^n . \sum_{n=1}^{\infty} \frac{1}{n!} s^{n-1} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} s^n . \sum_{n=0}^{\infty} \frac{1}{(n+1)!} s^n$$

$$1 = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\beta_m}{m!} \frac{1}{(n-m+1)} s^n = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{m=0}^{n} \binom{n+1}{m} \beta_m s^n$$

$$\sum_{m=0}^{n} \binom{n+1}{m} \beta_m = 0, \;\; \beta_0 = 1 \;\; and \; \beta_1 = -\frac{1}{2}$$

by the means of (6) and (7), we have

$$\pi s \cot \pi s = i\pi s + \sum_{n=0}^{\infty} \frac{\beta_n}{n!} (2i\pi s)^n$$

$$= i\pi s + \frac{\beta_0}{0!} + \frac{\beta_1}{1!}(2i\pi s) - 2\sum_{n=2}^{\infty} \frac{\beta_n}{n!}\left(-\frac{1}{2}\right)(2i\pi s)^n$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!}\left(-\frac{1}{2}\right)(2i\pi s)^{2n}$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{\beta_{2n}}{(2n)!}(-1)\frac{(2\pi)^{2n} i^{2n}}{2} s^{2n}$$

$$= 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2\pi)^{2n}\beta_{2n}}{2.(2n)!} s^{2n}$$

the expression above together with theorem 3.1 implies:

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}\beta_{2n}}{2.(2n)!}$$

□

***Examples:***

$$\zeta(2) = \frac{4\pi^2}{2.2!}\,\frac{1}{6} = \frac{\pi^2}{6}, \zeta(4) = -\frac{16\pi^4}{2.24}\,\frac{-1}{30} = \frac{\pi^4}{90} \; and \; \zeta(6) = \frac{32\pi^6}{2.720}\,\frac{1}{42} = \frac{\pi^6}{945}$$

## V. SPECIAL FUNCTIONS

***Theorem 5.1*”** Gamma Function” [3]**,** [4]

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{u^{s-1}}{e^u - 1} du$$

***Proof***. $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt, \, let \; t = nu \; \rightarrow dt = ndu$

$$\Gamma(s) = \int_0^{\infty} n^s u^{s-1} e^{-nu} \, du$$

$$\Gamma(s)\ \frac{1}{n^s} = \int_0^\infty u^{s-1} e^{-nu}\, du$$

$$\Gamma(s)\zeta(s) = \Gamma(s) \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \int_0^\infty u^{s-1} e^{-nu}\, du = \int_0^\infty u^{s-1} \sum_{n=1}^{\infty} (e^{-u})^n$$

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{u^{s-1}}{e^u - 1} du$$

□

***Theorem 5.2*** "Jacobi Theta Function"

$$\vartheta(x) = \frac{1}{\sqrt{x}} \vartheta\left(\frac{1}{x}\right)$$

***Proof.***

$$\vartheta(x) = \sum_{n \in Z} e^{-\pi n^2 x}$$

$$\sum_{n \in Z} f(n) = \sum_{k \in Z} \int_{-\infty}^{+\infty} f(y) e^{-2\pi iky}\, dy$$

$$\sum_{n \in Z} e^{-\pi n^2 x} = \sum_{k \in Z} \int_{-\infty}^{+\infty} e^{-\pi y^2 x}\ e^{-2\pi iky}\, dy = \sum_{k \in Z} \int_{-\infty}^{+\infty} e^{-\pi y^2 x - 2\pi iky}\, dy$$

$$= \sum_{k \in Z} \int_{-\infty}^{+\infty} e^{-\pi x\left(y^2 + 2i\frac{k}{x}y + i^2\frac{k^2}{x^2} - i^2\frac{k^2}{x^2}\right)} dy$$

$$= \sum_{k \in Z} \int_{-\infty}^{+\infty} e^{-\pi x\left[\left(y + i\frac{k}{x}\right)^2 - i^2\frac{k^2}{x^2}\right]} dy$$

$$= \sum_{k \in Z} \int_{-\infty}^{+\infty} e^{-\pi k^2 \frac{1}{x}} e^{-\pi x\left(y + i\frac{k}{x}\right)^2} dy$$

$$= \sum_{k \in Z} e^{-\pi k^2 \frac{1}{x}} \int_{-\infty}^{+\infty} e^{-\pi x\left(y + i\frac{k}{x}\right)^2} dy$$

let $y + i\ \frac{k}{x} = z \xrightarrow{yields} dy = dz\ \ \Big|_{-\infty}^{+\infty} \to \Big|_{-\infty + i\frac{k}{x}}^{+\infty + i\frac{k}{x}}$

$$\sum_{n \in Z} e^{-\pi n^2 x} = \sum_{k \in Z} e^{-\pi k^2 \frac{1}{x}} \int_{-\infty + i\frac{k}{x}}^{-\infty + i\frac{k}{x}} e^{-\pi x z^2} dz$$

now,

$$\int_{-R}^{R} e^{-\pi x z^2} dz = \int_{-R}^{-R + i\frac{k}{x}} e^{-\pi x z^2}\, dz + \int_{-R + i\frac{k}{x}}^{R + i\frac{k}{x}} e^{-\pi x z^2} dz + \int_{R + i\frac{k}{x}}^{R} e^{-\pi x z^2} dz$$

$$\sum_{n \in Z} e^{-\pi n^2 x} = \sum_{k \in Z} e^{-\pi k^2 \frac{1}{x}} \int_{-\infty}^{-\infty} e^{-\pi x z^2} dz = \sum_{k \in Z} e^{-\pi k^2 \frac{1}{x}} \sqrt{\frac{\pi}{\pi x}}$$

finally, we have

$$\sum_{n \in Z} e^{-\pi n^2 x} = \sum_{k \in Z} e^{-\pi k^2 \frac{1}{x}} \sqrt{\frac{1}{x}}$$

that is equivalent to:

$$\vartheta(x) = \frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right)$$

□

## VI. THE ZETA FUNCTIONAL EQUATIONS

***Theorem 6.1***

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

***Proof.*** In the definition of Gamma Function, if we let $\to \frac{s}{2}$ :

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt, \quad let\ t = \pi n^2 x \to dt = \pi n^2 dx:$$

$$= \int_0^\infty (\pi n^2 x)^{\frac{s}{2}-1}e^{-\pi n^2 x}\pi n^2 dx = \int_0^\infty \pi^{\frac{s}{2}} n^s x^{\frac{s}{2}-1} e^{-\pi n^2 x} dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\sum_{n=1}^\infty \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \sum_{n=1}^\infty\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^\infty\frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty e^{-\pi n^2 x}dx$$

recall:

$$\vartheta(x) = \sum_{n\in Z} e^{-\pi n^2 x} = 1 + 2\sum_{n=1}^\infty e^{-\pi n^2 x} = 1 + 2\psi(x)$$

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx = \int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx + \int_0^1 x^{\frac{s}{2}-1}\psi(x)dx$$

by using:

$$\vartheta(x) = \frac{1}{\sqrt{x}}\vartheta\left(\frac{1}{x}\right) \leftrightarrow 2\psi(x) + 1 = \frac{1}{\sqrt{x}}\left(2\psi\left(\frac{1}{x}\right) + 1\right)$$

$$\psi(x) = \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}$$

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx = \int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right)dx$$

$$= \int_0^1 \left\{x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right) + \frac{1}{2}\left(x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1}\right)\right\}dx$$

$$= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{2}\int_0^1\left(x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1}\right)dx$$

$$= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{2}\left\{\frac{1}{\frac{s}{2}-\frac{1}{2}}x^{\frac{s}{2}-\frac{1}{2}} - \frac{1}{\frac{s}{2}}x^{\frac{s}{2}}\right\}_0^1$$

$$= \int_0^1 x^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{x}\right)dx + \frac{1}{s(s-1)}$$

$$for\ x = \frac{1}{u} \to dx = -\frac{1}{u^2}du, \qquad |_0^1 \to |_\infty^1$$

the right hand side of the integration becomes:

$$\int_{\infty}^{1}\left(\frac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}}\psi(u)\left(-\frac{du}{u^2}\right)+\frac{1}{s(s-1)}$$

that is:

$$\int_{1}^{\infty}\left(\frac{1}{x}\right)^{\frac{s}{2}-\frac{3}{2}}\psi(x)\left(\frac{dx}{x^2}\right)+\frac{1}{s(s-1)}$$

we derive:

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx=\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx+\frac{1}{s(s-1)}$$

now,

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx=\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx+\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx$$

$$=\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx+\int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}}\psi(x)dx+\frac{1}{s(s-1)}$$

$$=\int_1^\infty\left(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(x)dx+\frac{1}{s(s-1)}$$

therefore; since

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}dx=\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

we have:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_1^\infty\left(x^{\frac{s}{2}}+x^{\frac{1-s}{2}}\right)\frac{\psi(x)}{x}dx-\frac{1}{s(1-s)}$$

this implies:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

□

***Theorem 6.2***

$$\zeta(s)=2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$$

***Proof***: by using:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

together with the following facts:

$$\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)=\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)=\frac{\pi}{\cos\frac{\pi s}{2}}$$

$$\Gamma(2s)=\frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$$

let $s\to\frac{s}{2}$ in the expression above, we have:

$$\Gamma(s)=\frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

that is:

$$\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)=\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\qquad(8)$$

we also know that:

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s}$$

if we let $s\to\frac{s+1}{2}$ we derive:

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right)}$$

or

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)=\frac{\pi}{\cos\left(\frac{\pi s}{2}\right)} \qquad (9)$$

we multiply both side of

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

by $\Gamma\left(\frac{s+1}{2}\right)$ implies:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

using (8) and (9):

$$\pi^{-\frac{s}{2}}\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)\zeta(s)=\pi^{-\frac{1-s}{2}}\frac{\pi}{\cos\frac{\pi s}{2}}\zeta(1-s)$$

that is

$$\zeta(1-s)=\frac{2}{(2\pi)^{s}}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s)$$

Letting *1-s* goes to *s*

$$\zeta(s)=\frac{2}{(2\pi)^{1-s}}\cos\frac{\pi(1-s)}{2}\Gamma(1-s)\zeta(1-s)$$

that completes the proof □

## VII. CONCLUSION

### A. THE TRIVIAL ZEROES

It is obvious from Theorem the result above that the following:

$$-2,-4,-6,\ldots=-2k, k=1,2,\ldots \textit{ are Zeroes of Zeta Function.}$$

The above zeroes are simply called Trivial Zeroes.

### B. RIEMANN Xi FUNCTION

Recall:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_{1}^{\infty}\left(x^{\frac{s}{2}}+x^{\frac{1-s}{2}}\right)\frac{\psi(x)}{x}dx-\frac{1}{s(1-s)}$$

Now, multiply both sides by $\frac{1}{2}s(s-1)$, we have:

$$\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{2}s(s-1),\int_{1}^{\infty}\left(x^{\frac{s}{2}}+x^{\frac{1-s}{2}}\right)\frac{\psi(x)}{x}dx-\frac{1}{2}$$

let

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

the first initial observation we have

$$\xi(s)=\xi(1-s)$$

if we choose $s=\frac{1}{2}+it$ the following holds true:

$$\xi\left(\frac{1}{2}+it\right)=\xi\left(1-\left(\frac{1}{2}+it\right)\right)$$

that is

$$\xi\left(\frac{1}{2}+it\right)=\xi\left(\frac{1}{2}-it\right)$$

that shows the symmetry of $\xi-$ Function with $x=\frac{1}{2}$ as the line of symmetry, and leads to the conclusion that if $s=x+iy$ is a root of Zeta function then $1-s=1-x+iy$ and $\bar{s}=x-iy$ are also zeroes of zeta function [5], [6] and [7]

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