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Maximum Entropy Analysis of M^x/G/1 Queue With Modified Bernoulli Vacation Schedule And Un-Reliable Server Under N-Policy

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ABSTARCT:- This paper examines the steady state behavior of unreliable server in a batch arrival queue with two phases of heterogeneous service along with Bernoulli schedule and N-policy. The server provides two kinds of services in succession, the first stage service (FSS) followed by second stage service (SSS). As soon as both services are completed, the server may take a vacation or may continue staying in the system. The server may break down during the service; the failure and repair times are assumed to follow a general distribution. By using the maximum entropy method (MEM), the approximate formulae for the probability distributions of the number of customers in the system have been derived, which are further used to obtain various system performance measures. A comparative analysis between approximate results established and exact results has also been performed. It is noticed that the maximum entropy approach provides reasonably good approximate solutions for practical purpose.

Key words: *Maximum entropy, Bernoulli vacation schedule, Batch arrival, N-policy, Two phase services, Un-reliable server.*

I. INTRODUCTION

Information theory provides a constructive criterion for setting up probability distribution on the basis of partial knowledge, called the maximum entropy estimate. It is least biased estimate, possible on the given information. Jain and Singh (2000) employed MEM to analyze the optimal flow control of G/G/c finite capacity queue via diffusion process. Herrero (2002) proposed a direct method to compute the second moment and also for probability of customers being served in a busy period. The methodology of maximum entropy has been used by Guan et al. (2009) to characterize closed form expression for the state and blocking probabilities for threshold based discrete time queue.Wang et al. (2002) analyzed the N-policy M/G/1 queueing system with removable server by using maximum entropy. Jain and Dhakad (2003a) provided the steady state queue size distribution for G/G/1 queue by using maximum entropy approach. Moreover, Jain and Dhakad (2003b) provided the steady state queue size distribution for M^x/G/1 queueing system with server breakdowns and general setup times. Entropy maximization and queuing network with priorities and blocking have been analyzed by Kouvatsos and Awan (2003). Maximum entropy principle is used by Wang et al. (2005), to derive approximate formulae for the steady state probability distribution of the queue length. Recently, Jain and Jain (2006) obtained approximate results for the queue size distribution for G/G/1 queue with vacation under N-policy. A comparative analysis has been made between the approximate results and exact results for M^x/G/1 queueing system with server vacation by Ke and Lin (2006). A single unreliable server M^X/G/1 queueing system with multiple vacations was considered by Wang et al. (2007). Wang and Huang (2009) analyzed a single removable and unreliable server queue and use maximum entropy approach to develop the approximate formulae for waiting time in the system.

In this investigation, we have derived the approximate formulae for the expected system size and expected waiting time for an M^x/G/1 queueing model with Bernoulli schedule vacations and repairable server. The organization of the paper is as follows. The model under consideration is described along with assumptions and limiting probabilities and governing equations in section 2. Various queueing characteristics along with queue length have been derived in section 3. Section 4 is devoted to Maximum Entropy Method (MEM). The approximate results for the expected queue size and expected waiting time are established in sections 5 and 6, respectively. A comparative analysis between the approximate and exact results is also performed. Numerical illustrations have been provided in next section 7. In the last section 8, the conclusions are made.

II. MODELS DESCRIPTION

We consider an M^X/G/1 queueing system with the following assumptions:

- The customers arrive at the system according to a compound Poisson process with random batch size denoted by random variable ‘X’.
- When N customers are accumulated in the system, the server starts service to the customers.
- There is a single unreliable server which provides two kinds of general heterogeneous services in the sequence to the customers on a first come first served (FCFS) basis i.e. first stage service (FSS) followed by the second stage service (SSS).
- As soon as the service of a customer is completed, the server may take a vacation with probability r or else with probability (1-r) he may continue servicing the next customer, if any, otherwise the system is turned off.
- We assume that the service time random variable S_i (i=1,2) of the i^{th} service follows a general probability law with $S_i(x)$ as the distribution function. We denote Laplace Stieltjes Transform (LST) of $S_i(x)$ by $S_i^*(s)$ with finite moment $E(S_i^k)$, $k\geq 1$, i=1,2.
- The vacation time V of the server follows a general probability law with distribution function V(x), LST V*(s) and finite moment $E(V^k)$, k=1,2.
- The server may breakdown while servicing the customers. When the server breaks down, it is sent for repair. The repair times are distributed with probability distribution function $G_i(x)$, i=1,2 while server fails during i^{th} phase service. Immediately after the server is repaired, it starts serving the customers.

Some other notations used for modeling purpose are as follows:

λ	Arrival rate of the customers
α_i	Failure rate of the server while rendering i^{th} (i=1,2) phase service
$\mu_i(x)$	Service rate of the server for i^{th} (i=1,2) phase service
$v(x)$	Vacation rate
$\beta_i(y)$	Repair rate of the server while broken down during i^{th} (i=1,2) phase service
$V(x)$, $v(x)$	Cumulative distribution function and density function of vacation time
$B_i(x)$, $b_i(x)$	Cumulative distribution function and density function of service time (i=1,2), respectively
$G_i(x)$, $g_i(x)$	Cumulative distribution function and density function of repair time (i=1,2), respectively
$V^*(s)$	LST of V(x)
$B^*_i(s)$	LST of $B_i(x)$
$G_i^*(s)$, $g_i(s)$	LST of $G_i(x)$, $g_i(x)$

Hazard rates are defined as

$$v(x) = \frac{dV(x)}{1 - V(x)}, \quad \mu_i(x) = \frac{dB_i(x)}{1 - B_i(x)}, \ (i = 1,2), \quad \beta_i(x) = \frac{dG_i(x)}{1 - G_i(x)}, \ (i = 1,2)$$

III. STEADY STATE EQUATIONS

The Chapman Kolmogorov equations are constructed as follows:

$$\lambda I_0 = \int_0^\infty v(x) Q_0(x)\,dx + (1 - r)\int_0^\infty \mu_2(x) P_{2,1}(x)\,dx \tag{1}$$

$$\lambda I_n = \lambda \sum_{i=1}^{n} a_i I_{n-i}, \qquad n = 1,2,\ldots, N-1 \tag{2}$$

$$\frac{d}{dx} P_{1,n}(x) + (\lambda + \mu_1(x) + \alpha_1) P_{1,n}(x) = \lambda \sum_{i=1}^{n} a_i P_{1,n-i}(x) + \int_0^\infty R_{1,n}(x,y)\beta_1(y)\,dy, \quad n \geq 1 \tag{3}$$

$$\frac{d}{dx} P_{2,n}(x) + (\lambda + \mu_2(x) + \alpha_{21}) P_{2,n}(x) = \lambda \sum_{i=1}^{n} a_i P_{2,n-i}(x) + \int_0^\infty R_{2,n}(x,y)\beta_{21}(y)\,dy, \quad n \geq 1 \tag{4}$$

$$\frac{d}{dx} R_{1,n}(x,y) + (\lambda + \beta_1(y)) R_{1,n}(x,y) = \lambda \sum_{i=1}^{n} a_i R_{1,n-i}(y) \tag{5}$$

$$\frac{d}{dx} R_{2,n}(x,y) + (\lambda + \beta_2(y)) R_{2,n}(x,y) = \lambda \sum_{i=1}^{n} a_i R_{2,n-i}(y) \tag{6}$$

$$\frac{d}{dx} Q_n(x) + (\lambda + v(x)) Q_n(x) = \lambda \sum_{i=1}^{n} a_i Q_{n-i}(x), \qquad n \geq 1 \tag{7}$$

$$\frac{d}{dx}Q_0(x) + (\lambda + v(x))\,Q_0(x) = 0, \qquad n = 0 \tag{8}$$

These equations are to be solved subject to the following boundary conditions:

$$P_{1,n}(0) = (1-r)\int_0^\infty \mu_2(x)\,P_{2,n+1}(x)dx + \int_0^\infty v(x)\,Q_n(x)\,dx, \quad n = 1,2,\ldots, N-1 \tag{9}$$

$$P_{1,n}(0) = (1-r)\int_0^\infty \mu_2(x)\,P_{2,n+1}(x)dx + \int_0^\infty v(x)\,Q_n(x)\,dx + \lambda\sum_{i=0}^{n} a_i I_{n-i}, \quad n \ge N \tag{10}$$

$$P_{2,n}(0) = \int_0^\infty \mu_1(x)\,P_{1,n}(x)dx, \quad n \ge 1 \tag{11}$$

$$Q_n(0) = r\int_0^\infty \mu_2(x)\,P_{2,n+1}(x)dx, \quad n \ge 0 \tag{12}$$

$$R_{1,n}(x,0) = \alpha_1 P_{1,n}(x), \quad n \ge 1 \tag{13}$$

$$R_{2,n}(x,0) = \alpha_2 P_{2,n}(x), \quad n \ge 1 \tag{14}$$

The normalizing equation is given by

$$\sum_{n=0}^{N-1} I_n + \sum_{i=1}^{2}\sum_{n=1}^{\infty}\int_0^\infty P_{i,n}(x)dx + \sum_{i=1}^{2}\sum_{n=1}^{\infty}\int_0^\infty\int_0^\infty R_{i,n}(x,y)dx\,dy + \sum_{n=0}^{\infty}\int_0^\infty Q_n(x)dx = 1 \tag{15}$$

Now we define some probability generating functions as follows:

$P_i(x;z) = \sum_{n=1}^{\infty} z^n P_{i,n}(x),\ i = 1,2$; $R_i(x;y;z) = \sum_{n=1}^{\infty} z^n R_{i,n}(x,y),\ i = 1,2$; $Q(x;z) = \sum_{n=0}^{\infty} z^n Q_n(x)$

Also $P_i(z) = \int_0^\infty P_i(x,z)dx,\ i = 1,2$; $R_i(z) = \int_0^\infty\int_0^\infty R_i(x,y,z)dxdy,\ i = 1,2$; $Q(z) = \int_0^\infty Q(x,z)dx$

IV. THE ANALYSIS

Theorem 1: The partial joint generating functions when the server is busy with FSS and SSS, under breakdown while rendering service during FSS and SSS and on vacations respectively, are obtained by solving equations (1)- (14)

$$P_1(x,z) = P_1(0,z)\exp[\{-\xi_1(z)\}x]\{1 - B_1(x)\} \tag{16}$$

$$P_2(x,z) = P_2(0,z)\exp[\{-\xi_2(z)\}x]\{1 - B_2(x)\} \tag{17}$$

$$R_1(x,y,z) = R_1(x,0,z)\exp[\{-\lambda(1 - X(z)\}x]\{1 - G_1(y)\} \tag{18}$$

$$R_2(x,y,z) = R_2(x,0,z)\exp[\{-\lambda(1 - X(z)\}x]\{1 - G_2(y)\} \tag{19}$$

$$Q(x,z) = Q(0,z)\exp[\{-\lambda(1 - X(z)\}x]\{1 - V(x)\} \tag{20}$$

where $\xi_i(z) = \lambda + \alpha_i - \lambda X(z) - \alpha_i g_i{}^*(\lambda - \lambda X(z))$

Proof: For proof see appendix *A-I.*

Theorem 2: The marginal generating functions are

$$P_1(z) = P_1(0,z)\left\{\frac{1 - b^*(\xi_1(z))}{\xi_1(z)}\right\} \tag{21}$$

$$P_2(z) = P_1(0,z)b^*(\xi_1(z))\left\{\frac{1 - b^*(\xi_2(z))}{\xi_2(z)}\right\} \tag{22}$$

$$R_1(z) = \alpha_1 P_1(0,z)\left\{\frac{1-b^*(\xi_1(z))}{\xi_1(z)}\right\}\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \tag{23}$$

$$R_2(z) = \alpha_2 P_1(0,z) b^*(\xi_1(z))\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\}\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \tag{24}$$

$$Q(z) = rP_1(0,z) b^*(\xi_1(z) b^*(\xi_2(z)) z^{-1}\left\{\frac{1-v^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \tag{25}$$

where

$$P_1(0,z) = \frac{z\lambda I(z)\{X(z)-1\}}{z-\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))} \tag{26}$$

and $I(z) = \dfrac{(1-\phi)\sum_{n=0}^{N-1} z^n \psi_n}{\sum_{n=0}^{N-1}\psi_n}$

$\phi = \lambda E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}$

Proof: See appendix *A-II*.

Theorem 3: The probability generating function of the customers in the queue is given by

$$P(z) = \frac{(1-z)I(z)\{1-r+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))}{\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))-z} \tag{27}$$

Proof: See appendix *A-III* for proof.

Corollary: If the system is in steady state, then

Prob[The server is in idle state]=I(1)=1-λE[X]{rE[V]-E[B_1](1+α_1E[G_1]) -E[B_2](1+α_2E[G_2]) (28)

Prob[The server is busy with FSS]=P_1(1)=λE[X]E[S_1] (29)
Prob[The server is busy with SSS]=P_2(1)=λE[X]E[S_2] (30)
Prob[The server is broken down while FSS]=R_1(1)=$\alpha_1\lambda$E[X]E[G_1] (31)
Prob[The server is broken down while SSS]=R_2(1)=$\alpha_2\lambda$E[X]E[G_2] (32)
Prob[The server is on vacation]=Q(1)=rλE[X]E[V] (33)

Theorem 4: The expected number of customers in the system is

$$L = \left.\frac{dP(z)}{dz}\right|_{z=1}$$

$$= \phi + \frac{\phi\lambda E(X(X-1))}{2(1-\phi)} + \frac{\lambda rE(X)E^2(V)}{2(1-\phi)} - \frac{\{E^2(B_1)(\alpha_1 E(G_1)+1)+E^2(B_2)(\alpha_2 E(G_2)+1)\}}{2(1-\phi)}$$
$$- \frac{\{\alpha_1 E(B_1)E^2(G_1)+\alpha_2 E(B_2)E^2(G_2)\}}{2(1-\phi)} - \frac{r\lambda^2 E^2(X)E(V)\{E(B_1)(\alpha_1 E(G_1)+1)+E(B_2)(\alpha_2 E(G_2)+1)\}}{(1-\phi)} \tag{34}$$

where $\phi = \lambda E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}$.

Proof: The number of customers in the system can be obtained by using $L = \lim_{z\to 1} P'(z)$.

L'Hospital rule is used repetitively to compute L.

V. PRINCIPLE OF MAXIMUM ENTROPY

In this section maximum entropy model is formulated in order to develop the steady state probabilities, by applying the method of entropy maximization. The steady state probabilities are defined as follows:

I(n): The probability that server is idle

P_i(n): The probability that there are n customers in the system and server provides i^{th} (i=1,2) stage service

R_i(n): The probability that there are n customers in the system and server is under broken down state while providing i^{th} (i=1,2) stage service

Q(n): The probability that the server is on vacation

The entropy function Y can be formulated mathematically as

$$Y = -\sum_{n=0}^{N-1} I(n)\log I(n) - \sum_{n=1}^{\infty} P_1(n)\log P_1(n) - \sum_{n=1}^{\infty} P_2(n)\log P_2(n) - \sum_{n=1}^{\infty} R_1(n)\log R_1(n)$$
$$- \sum_{n=1}^{\infty} R_2(n)\log R_2(n) - \sum_{n=1}^{\infty} Q(n)\log Q(n) \tag{35}$$

The maximum entropy solution for the $M^x/G/1$ queueing system is obtained by maximizing (34) subject to the following constraints:

$$\sum_{n=1}^{N-1} I(n) = 1 - \lambda E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\} \tag{36}$$

$$\sum_{n=1}^{\infty} P_i(n) = \lambda E[X]E[B_i] \tag{37}$$

$$\sum_{n=1}^{\infty} R_i(n) = \alpha_i \lambda E[X]E[G_i],\ i = 1,2 \tag{38}$$

$$\sum_{n=1}^{\infty} Q(n) = r\lambda E[X]E[V] \tag{39}$$

Expected number of customers in the system is given by L, as follows

$$\sum_{n=1}^{\infty} n\{I(n) + P_1(n) + P_2(n) + R_1(n) + R_2(n) + Q(n)\} = L \tag{40}$$

where L is given by equation (34).

Multiplying (36)-(39) by Lagrangian multipliers $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$ respectively, and the Lagrangian function $H\{I(n), P_1(n), P_2(n), R_1(n), R_2(n), Q(n)\}$ can be obtained as

$$H\{P_1(n), P_2(n), R_1(n), R_2(n), Q(n)\} = -\sum_{n=0}^{N-1} I(n)\log I(n) - \sum_{n=1}^{\infty} P_1(n)\log P_1(n) - \sum_{n=1}^{\infty} P_2(n)\log P_2(n) - \sum_{n=1}^{\infty} R_1(n)\log R_1(n) - \sum_{n=1}^{\infty} R_2(n)\log R_2(n)$$
$$- \sum_{n=1}^{\infty} Q(n)\log Q(n) - \theta_1\left\{\sum_{n=0}^{N-1} I(n) - 1 + \lambda E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}\right\} - \theta_2\left\{\sum_{n=1}^{\infty} P_1(n) - \lambda E[X]E[B_1]\right\}$$
$$- \theta_3\left\{\sum_{n=1}^{\infty} P_2(n) - \lambda E[X]E[B_2]\right\} - \theta_4\left\{\sum_{n=1}^{\infty 1} R_1(n) - \alpha_1\lambda E[X]E[G_1]\right\} - \theta_5\left\{\sum_{n=1}^{\infty} R_2(n) - \alpha_2\lambda E[X]E[G_2]\right\} - \theta_6\left\{\sum_{n=1}^{\infty} Q(n) - r\lambda E[X]E[V]\right\}$$
$$- \theta_7\left\{\sum_{n=0}^{N-1} nI(n) + \sum_{n=1}^{\infty} n\big(P_1(n) + P_2(n) + R_1(n) + R_2(n) + Q(n)\big) - L\right\} \tag{41}$$

VI. MAXIMUM ENTROPY SOLUTION

To find maximum entropy solutions I(n), P_1(n), P_2(n), R_1(n), R_2(n), Q(n), maximizing in (19) subject to constraint (20)-(25) is equivalent to maximizing (26). The maximum entropy solutions can be obtained by taking partial derivatives of H with respect to I(n) P_1(n), P_2(n), R_1(n), R_2(n), Q(n) and setting the results equal to zero. Now

$$\frac{\partial H}{\partial I(n)} = -\log \ I(n) - 1 - \theta_1 - n\theta_7 = 0 \quad (42)$$

$$\frac{\partial H}{\partial P_1(n)} = -\log \ P_1(n) - 1 - \theta_2 - n\theta_7 = 0 \quad (43)$$

$$\frac{\partial H}{\partial P_2(n)} = -\log \ P_2(n) - 1 - \theta_3 - n\theta_7 = 0 \quad (44)$$

$$\frac{\partial H}{\partial R_1(n)} = -\log \ R_1(n) - 1 - \theta_4 - n\theta_7 = 0 \quad (45)$$

$$\frac{\partial H}{\partial R_2(n)} = -\log \ R_2(n) - 1 - \theta_5 - n\theta_7 = 0 \quad (46)$$

$$\frac{\partial H}{\partial Q(n)} = -\log \ Q(n) - 1 - \theta_6 - n\theta_7 = 0 \quad (47)$$

Therefore

$$I(n) = e^{-(1+\theta_1)} e_7^{-n\theta_7} \quad (48)$$

$$P_1(n) = e^{-(1+\theta_2)} e_7^{-n\theta_7} \quad (49)$$

$$P_2(n) = e^{-(1+\theta_3)} e_7^{-n\theta_7} \quad (50)$$

$$R_1(n) = e^{-(1+\theta_4)} e_7^{-n\theta_7} \quad (51)$$

$$R_2(n) = e^{-(1+\theta_5)} e_7^{-n\theta_7} \quad (52)$$

$$Q(n) = e^{-(1+\theta_6)} e_7^{-n\theta_7} \quad (53)$$

Let $\psi_i = e^{-(1+\theta_i)}$, i=1,2,…,6 and $\psi_7 = e^{-\theta_7}$.
By using this in equations (48)-(53), we obtain

$$I(n) = \psi_1\psi_7^n ; P_1(n) = \psi_2\psi_7^n ; P_2(n) = \psi_3\psi_7^n ; R_1(n) = \psi_4\psi_7^n ; R_2(n) = \psi_5\psi_7^n ; Q(n) = \psi_6\psi_7^n \quad (54)$$

Substituting (50) in (35)-(37), we obtain

$$\frac{\psi_1\psi_7}{1-\psi_7} = 1 - \lambda E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\} \quad (55)$$

$$\frac{\psi_2\psi_7}{1-\psi_7} = \lambda E[X]E[B_1] \quad (56)$$

$$\frac{\psi_3\psi_7}{1-\psi_7} = \lambda E[X]E[B_2] \quad (57)$$

$$\frac{\psi_4\psi_7}{1-\psi_7} = \alpha_1\lambda E[X]E[R_1] \quad (58)$$

$$\frac{\psi_5\psi_7}{1-\psi_7} = \alpha_2\lambda E[X]E[R_1] \quad (59)$$

$$\frac{\psi_6\psi_7}{1-\psi_7} = r\lambda E[X]E[V] \quad (60)$$

$$L = \frac{\rho}{(1-\psi_6)} \quad (61)$$

where $\rho = \lambda E[X]\{E[B_1] + E[B_2] + \alpha_1 E[G_1] + \alpha_2 E[G_2] + rE[V]\}$.
From (61), we obtain

$$\psi_7 = \frac{L-\rho}{L} \quad (62)$$

After algebraic manipulation we obtain ψ_i (i=1,2,3,4,5,6) as follows:

$$\psi_1 = \frac{\lambda\rho E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}}{(L-\rho)} \tag{63}$$

$$\psi_2 = \frac{\lambda\rho E[X]E[B_1]}{(L-\rho)} \tag{64}$$

$$\psi_3 = \frac{\lambda\rho E[X]E[B_2]}{(L-\rho)} \tag{65}$$

$$\psi_4 = \frac{\alpha_1\lambda\rho E[X]E[G_1]}{(L-\rho)} \tag{66}$$

$$\psi_5 = \frac{\alpha_2\lambda\rho E[X]E[G_2]}{(L-\rho)} \tag{67}$$

$$\psi_6 = \frac{r\lambda\rho E[X]E[V]}{(L-\rho)} \tag{68}$$

Substituting the value of ψ_i, i=1,2,3,4,5,6 in (54), we get

$$I(n) = \frac{\lambda\rho E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}(L-\rho)^{n-1}}{L^n} \tag{69}$$

$$P_1(n) = \frac{\lambda\rho E[X]E[B_1](L-\rho)^{n-1}}{L^n} \tag{70}$$

$$P_2(n) = \frac{\lambda\rho E[X]E[B_2](L-\rho)^{n-1}}{L^n} \tag{71}$$

$$R_1(n) = \frac{\alpha_1\lambda\rho E[X]E[G_1](L-\rho)^{n-1}}{L^n} \tag{72}$$

$$R_2(n) = \frac{\alpha_2\lambda\rho E[X]E[G_2](L-\rho)^{n-1}}{L^n} \tag{73}$$

$$Q(n) = \frac{r\lambda\rho E[X]E[V](L-\rho)^{n-1}}{L^n} \tag{74}$$

VII. EXPECTED WAITING TIME IN THE QUEUE

In this section, we derive the exact and approximate solutions for the expected waiting time in the system, as follows:

(a) The exact waiting time in the queue:

The exact expected waiting time (W) can be obtained using Little's formula, as

$W = \dfrac{L}{\lambda E[X]}$, where L is given in equation (33).

(b) The approximate expected waiting time in the queue:

The approximate expected waiting time $\hat{W}$ can be obtained as

$$\hat{W} = \sum_{n=0}^{N-1}\left(\frac{N-n-1}{\lambda} + E[B_1] + E[B_1]\right)I(n) + \sum_{n=1}^{\infty} nE[B_1]P_1(n) + \sum_{n=1}^{\infty} nE[B_2]P_2(n)$$
$$+\left(\sum_{n=1}^{\infty} nE[B_1] + \frac{E[G_1^2]}{2E[G_1]}\right)R_1(n) + \left(\sum_{n=1}^{\infty} nE[B_2] + \frac{E[G_2^2]}{2E[G_2]}\right)R_2(n) + rnE[V]Q(n) \tag{75}$$

By applying the value of $P_1(n)$, $P_2(n)$, $R_1(n)$, $R_2(n)$, $Q(n)$, we finally get

$$\hat{W} = \frac{\lambda LE[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}}{(L-\rho)}\left\{\frac{N-1}{\lambda}\right.$$
$$\left. + E[B_1] + E[B_2] - \frac{L}{\lambda\rho}\right\}\left\{1-\left(\frac{L-\rho}{L}\right)^N\right\} + \frac{\lambda LE[X]}{\rho}$$
$$\{E^2[B_1] + E^2[B_2] + \alpha_1 E[G_1]E[B_1] + \alpha_2 E[G_2]E[B_2] + rE^2[V]\}$$
$$+\frac{\lambda E[X]\{\alpha_1 E[G_1^2] + \alpha_2 E[G_2^2]\}}{2} \tag{76}$$

VIII. SENSITIVITY ANALYSIS

In order to determine the accuracy of approximate results obtained by using MEM, numerical comparisons between exact waiting time (W) and approximate waiting time ($\hat{W}$) has been done. We consider batch size to be geometrically distributed, whereas service time is assumed to be generally distributed. For computation purpose, we set the input data as given below:

Table 1: μ_1=6.0, μ_2=5.0, α_1=0.02, α_2=0.04, β_1=3.0, β_2=4.0, λ=0.3, r=0.2, v=0.4

Table 2: α_1=0.05, α_2=0.04, β_1=3.0, β_2=4.0, λ=0.4, r=0.7, v=0.4

Using the above data, we perform a numerical experiment analysis by developing computer program in software MATLAB to calculate the exact waiting time (W) and approximate waiting time ($\hat{W}$). All numerical results are summarized in table 1-2.

From table 1, it is seen that both W and $\hat{W}$ follow increasing trend with the increase in batch size. This is common phenomenon that can be seen in bulk queues. Table 2 displays a comparison of waiting times by varying the service rates μ_1 and μ_2. It is noticed that the increment of service rates results in decrease in waiting time for both exponential and gamma distributions. Relative percentage error for exponential distribution is 7-8% whereas 10-12% error has been noticed for gamma distribution.

IX. CONCLUDING REMARKS

Maximum entropy principle provides inverse methodology i.e. to evaluate queue size distribution in terms of known operational characteristics for analyzing the complex queueing systems. In this paper, we have applied maximum entropy method to develop closed form formulae for the probabilities and waiting time for an $M^x/G/1$ queueing system with Bernoulli schedule vacations with unreliable server under N-policy. We have performed a comparative analysis between the approximate results obtained using maximum entropy principle and exact results obtained using generating function approach.

X. APPENDIX

***A-I.* Proof of theorem 1:**

Multiply equations (3)-(7) and (13)-(14) with appropriate power of z and add:

$$\frac{d}{dx}P_1(x,z) + \left(\lambda + \mu(x) + \alpha_1 - \lambda X(z)\right)P_1(x,z) = \int_0^{\infty} R_1(x,y,z)\beta_1(y)dy \tag{A.1}$$

$$\frac{d}{dx}P_2(x,z)+\left(\lambda+\mu_2(x)+\alpha_2-\lambda X(z)\right)P_2(x,z)=\int_0^\infty R_2(x,y,z)\beta_2(y)dy \quad (A.2)$$

$$\frac{d}{dy}R_1(x,y,z)+\left(\lambda+\beta\ (y)-\lambda X(z)\right)R_1(x,y,z)=0 \quad (A.3)$$

$$\frac{d}{dy}R_2(x,y,z)+\left(\lambda+\beta_2(y)-\lambda X(z)\right)R_2(x,y,z)=0 \quad (A.4)$$

$$\frac{d}{dx}Q(x,z)+\left(\lambda+v(x)-\lambda X(z)\right)Q(x,z)=0 \quad (A.5)$$

$$R_1(x,0,z)=\alpha_1 P_1(x,z) \quad (A.6)$$

$$R_2(x,0,z)=\alpha_2 P_2(x,z) \quad (A.7)$$

Equation15-21 gives

$$P_1(x,z)=P_1(0,z)\exp[\{-\xi_1(z)\}x]\{1-B_1(x)\} \quad (A.8)$$

$$P_2(x,z)=P_2(0,z)\exp[\{-\xi_2(z)\}x]\{1-B_2(x)\} \quad (A.9)$$

$$R_1(x,y,z)=R_1(x,0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-G_1(y)\} \quad (A.10)$$

$$R_2(x,y,z)=R_2(x,0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-G_2(y)\} \quad (A.11)$$

$$Q(x,z)=Q(0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-V(x)\} \quad (A.12)$$

where $\xi_i(z)=\lambda+\alpha_i-\lambda X(z)-\alpha_i g_i*(\lambda-\lambda X(z))$

***A-II.* Proof of Theorem 2:**

Similarly equations (9) - (12) yields

$$P_1(0,z)=(1-r)P_2(0,z)z^{-1}b^*(\xi_2(z))+Q(0,z)v*(\lambda-\lambda X(z)) + \lambda I(z)(X(z)-1) \quad (A.13)$$

$$P_2(0,z)=P_1(0,z)b^*(\xi_1(z)) \quad (A.14)$$

$$Q(0,z)=rP_2(0,z)b^*(\xi_2(z))z^{-1} \quad (A.15)$$

Therefore by using equation (A.14) and (A.15) in (A.13)

$$P_1(0,z)=\frac{z\lambda I(z)\{X(z)-1\}}{z-\{(1-r)+rV*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))} \quad (A.16)$$

Integrating equation (A.8) with regard to z by parts and using equation (A.13) we get

$$P_1(z)=\int_0^\infty P_1(x,z)dx = P_1(0,z)\left\{\frac{1-b*(\xi_1(z))}{\xi_1(z)}\right\} \quad (A.17)$$

Similarly

$$P_2(z) = \int_0^\infty P_2(x,z)dx$$

$$= P_2(0,z)\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\} \tag{A.18}$$

$$= P_1(0,z)b^*(\xi_1(z))\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\}$$

$$Q(z) = \int_0^\infty Q(x,z)dx$$

$$= Q(0,z)\left\{\frac{1-v^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \tag{A.19}$$

$$= rP_1(0,z)b^*(\xi_1(z)b^*(\xi_2(z))z^{-1}\left\{\frac{1-v^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\}$$

$$R_1(z) = \int_0^\infty \int_0^\infty R_1(x,y,z)dx\,dy$$

$$= \alpha_1 P_1(z)\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \tag{A.20}$$

$$= \alpha_1 P_1(0,z)\left\{\frac{1-b^*(\xi_1(z))}{\xi_1(z)}\right\}\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\}$$

And

$$R_2(z) = \int_0^\infty \int_0^\infty R_2(x,y,z)dx\,dy$$

$$= \alpha_2 P_2(z)\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \tag{A.21}$$

$$= \alpha_2 P_1(0,z)b^*(\xi_1(z))\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\}\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\}$$

***A-III*. Proof of Theorem 3:**

PGF of the stationary queue size distribution at random epoch is given by

$$P(z) = I(z) + P_1(z) + P_2(z) + R_1(z) + R_2(z) + zQ(z) \tag{A.22}$$

Therefore

$$P(z) = P_1(0,z)\left\{\left\{\frac{1-b^*(\xi_1(z))}{\xi_1(z)}\right\} + b^*(\xi_1(z))\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\} + \alpha_1\left\{\frac{1-b^*(\xi_1(z))}{\xi_1(z)}\right\}\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} + \alpha_2 b^*(\xi_1(z))\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\}\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} + rb^*(\xi_1(z))b^*(\xi_2(z))z^{-1}\left\{\frac{1-v^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\}\right\} \quad \text{(A.23)}$$

Now substituting the value of $P_1(0,z)$ from equation (26) in equation (A.22), we get

$$P(z) = \frac{(1-z)I(z)\{1-r+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))}{\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))-z} \quad \text{(A.24)}$$

Let ψ_n (n=0,1,2,…,N-1) be the probability that a batch of customers finds at least n customers in the system during an idle period where ψ_n is given by the following recursive equation:

$$\psi_n = \sum_{i=1}^{n} a_i \psi_{n-i} \quad (n = 0,1,2,\ldots,(N-1)) \text{ and } \psi_0=1 \quad \text{(A.25)}$$

Now $I_n=I_0\,\psi_n$, where I_0 is normalizing constant, therefore

$$I(z) = I_0 \sum_{n=0}^{N-1} z^n \psi_n \quad \text{(A.26)}$$

To determine I_0, we use normalizing condition P(1)=1 and get
I(z)=(1-Φ), thus

$$I(z) = \frac{(1-\phi)\sum_{n=0}^{N-1} z^n \psi_n}{\sum_{n=0}^{N-1} \psi_n} \quad \text{(A.27)}$$

where $\Phi=r\lambda E[X]E[V]-\lambda E[B_1]E[X](1+\alpha_1E[G_1])-\lambda E[B_2]E[X](1+\alpha_2E[G_2])$ and $\sum_{n=0}^{N-1}\psi_n$ is the mean number of batches arriving during the idle period.

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E[X]	$M^x/M/1$			$M^x/\gamma/1$		
	W	$\hat{W}$	% error	W	$\hat{W}$	% error
5	2.15	2.08	3.25	11.89	10.87	8.59
6	2.17	2.09	3.68	12.75	11.63	8.78
7	2.20	2.11	4.03	14.15	13.63	3.67
8	2.23	2.12	4.93	15.30	14.45	5.55
9	2.25	2.14	4.94	16.01	15.19	5.12

Table 1: Effect of E[X] on W and $\hat{W}$ for $M^x/M/1$ and $M^x/\gamma/1$ models

(μ_1, μ_2)	$M^x/M/1$			$M^x/\gamma/1$		
	W	$\hat{W}$	% error	W	$\hat{W}$	% error
(6,4)	1.53	1.45	8.63	1.59	1.43	10.91
(8,4)	1.47	1.36	8.24	1.58	1.42	11.23
(8,6)	1.49	1.38	8.00	1.57	1.40	11.63
(6,8)	1.51	1.40	7.84	1.56	1.39	12.12
(8,8)	1.52	1.41	7.75	1.55	1.37	12.73

Table 2: Effect of (μ_1, μ_2) on W and $\hat{W}$ for $M^x/M/1$ and $M^x/\gamma/1$ models