IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 07, Issue 06 (June. 2017), ||V1|| PP 01-06

Total Bondage Number Of A Butterfly Graph

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ABSTRACT:- Domination Theory is an important branch of Graph Theory that has wide range of applications to various branches of Science and Technology. A new family of graphs called Butterfly Graphs is introduced recently and study of its parameters is under progress. Butterfly Graphs are undirected graphs and are widely used in interconnection networks. Let S be a subset of the set E of edges of G. Then the total bondage number $b_t(G)$ of G, is the minimum cardinality among all sets S such that $\gamma_t(G - S) > \gamma_t(G)$. In this paper the values for total bondage number of butterfly graph of dimension n are presented.

Key words: *Butterfly graph, Domination set, Total bondage number*
Subject Classification: 68R10

I. INTRODUCTION

A concept connected to domination numbers, called bondage number of a graph was studied by Fink, Jacobson, Kinch and Roberts [7]. Let S be a subset of the set E of edges of G. Then the total bondage number $b_t(G)$ of G, is the minimum cardinality among all sets S such that $\gamma_t(G - S) > \gamma_t(G)$. Now the values for total bondage number of butterfly graph of dimension n are presented. and some bounds on them are discussed.As dominating sets are required to study this concept, Chapters 4, 5 of [11] are referred , for the results on domination and total domination.

Lemma 1 : The total bondage number of BF(2) is 3.

Proof: Consider the graph BF(2). Then $\gamma_t(BF(2)) = 4$ (Lemma 5.1, Chapter 5 of [11]). By the definition of edges in BF(2), observe that a vertex (0; s) of L_0 is adjacent to a vertex (0; r) of L_1, for s, r = 0, 1, 2, 3, where $s + r \neq 3$. Let e be an edge joining the vertices (0; s) and (0; r) such that $s + r \neq 3$. Consider the graph BF(2) \ {e}. Let T = { (0; m_1), (0; m_2), (1; m_1), (1; m_2) / $\left| m_1 - m_2 \right| = 1$ or 2 or 3 }. The possible total dominating sets T in BF(2) are given below.

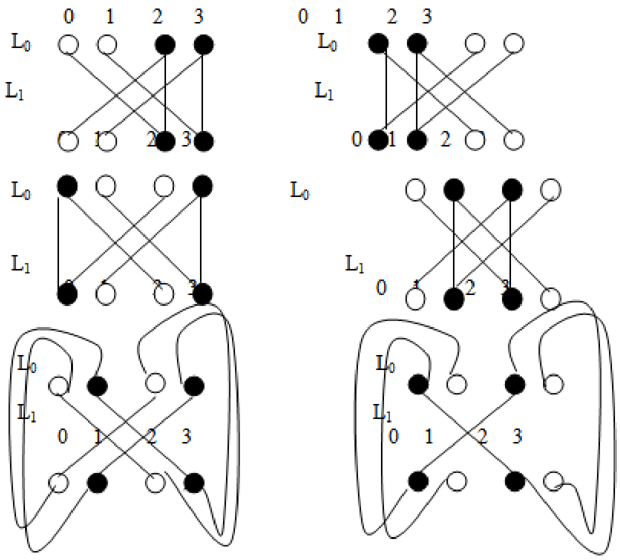
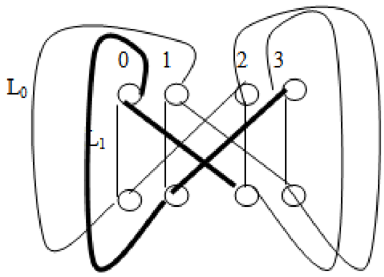


Fig.1

Let e_1, e_2 , e_3 be any three edges of BF(2), given by e_1 = {(o; r), (1; s)} , e_2 = {(o; t_1), (1; t_2)} , e_3 = {(1; p), (0; q)} , where $s + r \neq 3$, $t_1 + t_2 \neq 3$, $p + q \neq 3$. Let F = { e_1, e_2 , e_3 }. Consider the graph BF(2) \ F . Without loss of generality take r = 0 , s = 1 and t_1 = 0 and t_2 = 2 , p = 1 , q = 3 . Then e_1 = {(0; 0), (1; 1)}, e_2 = {(0;0), (1; 2)}, e_3 = {(1;1), (0; 3)}. Then for all possible choices of T given above , it can be verified that no T can dominate one of the end vertices of e_1 or e_2 or e_3 .

The edges in F are shown in bold.



Then consider the total dominating set T given by T = {(0; 0), (1;0), (0;3), (1;3)} . It is obvious that, this set can not dominate the vertex (1, 1) as e_1 and e_2 are the only edges joining (0;0) and (1; 1) and these are deleted. Therefore, adjoin (1;1) toT, so that it becomes T = {(0; 0), (1;0), (0;3), (1;3), (1;1)}. Now T dominates all vertices of BF(2) \ F . Further this set is also minimum. That is T is a minimum total dominating set of BF(2) \ F. Then γ_t (BF(2) \ F) = 5 .

Hence γ_t (BF(2) \ F) > γ_t (BF(2)).

Also observe that for all possible values of r , s, t_1 , t_2 , p , q such that s + r ≠ 3, $t_1 + t_2 \neq 3$, p + q ≠ 3 , any of the above mentioned total dominating sets can not dominate one of the end vertices of e_1 or e_2 or e_3. Hence for all possible choices of e_1, e_2, and e_3, γ_t (BF(2) \ F) > γ_t (BF(2)). Thus b_t(BF(2)) = $|F|$ = 3. □

Lemma 2 : The total bondage number of BF(3) is 6.

Proof : Consider the graph BF(3). We know that γ_t(BF(3)) = 8 (Lemma 5.2, Chapter 5 of [11]). The selection of vertices into a total dominating set of BF(3) is as follows.

Consider two adjacent vertices (k; r), (k + 1; s) , r, s = 1, 2…7, k = 0, 1, 2, and let e denote the edge joining these two vertices . There are seven edges incident on these two vertices. If all seven edges are deleted , then these vertices become isolated. As a total dominating set is required, at the most 6 edges incident on these two vertices can be deleted . Let F denote the set of edges incident on these two vertices except ‘e ‘ . Then $|F|$ = 6 .

Consider BF(3) \ F. Let T_1 denote a total dominating set of BF(3) \ F. Then include both the vertices (k; r) and (k + 1; s) into T_1 , as ‘ e ‘ is the only edge which dominates these two vertices in BF(3) \ F. As per the construction of a total dominating set of BF(3), include four vertices from left copy and four vertices from right copy into T_1. This selection of vertices does not include the vertices (k; r) and (k + 1; s) as they can not dominate any other vertex, except themselves. Since the domination is total, include these two vertices into T_1 so that $|T_1|$ = 10.

Hence γ_t(BF(3) \ F) > γ_t(BF(3)).

Now we claim that removal of any five edges from BF(3) does not increase the total domination number of BF(3). This is proved by taking the edges in $K_{2,2}$ s , between L_0 , L_1 .

Case 1 : Suppose F = {{ (0; r), (2; r) }, { (0; r), (2; r_1) }, { (0; r), (1; r + 2) }, { (0; s), (1; s + 2)}, { (0; s), (2; s)} } where r = 0, 1, and $|r - s|$ = 1, $|r - r_1|$ = 1. Consider the graph BF(3) \ F. The selection of vertices for domination in BF(3) \ F is as follows. Let T_1 denote a total dominating set of BF(3) \ F in the left copy.

Consider a $K_{2,2}$ between L_0 and L_1 incident on four vertices (0; r), (0; r + 2), (1; r), (1; r + 2) in the left copy where r = 0, 1. Since the edge incident on two vertices (0; r) , (1; r+2) is deleted , select the vertices (0; r + 2) and (1; r) into T_1 and these vertices are adjacent. The vertex (1; r) dominates the vertex (0; r) of L_0 and (2 ,r) of L_2 . The vertex (0, r+2) dominates the vertex (1,r+2 of L_1 and (2; r + 2) of L_2 .

Consider another $K_{2,2}$ between L_0 and L_1 incident on four vertices (0; s), (0; s + 2), (1; s), (1; s + 2) in the left copy where $|r - s|$ = 1. Since the edge incident on two vertices (0; s) , (1; s + 2) is deleted, select the vertices (0; s+2) and (1; s) which are adjacent, into T_1. The vertex (0; s+2) dominatesthevertex
(1; s + 2) of L_1 and (2; s + 2) of L_2 . The vertex (1; s) dominates the vertex (0; s) of L_0 and (2; s) of L_2 .

Then T_1 = {(0; r + 2), (0; s + 2) ,(1; r), (1; s)}.Clearly T_1 dominates all vertices in the left copy of BF(3) \ F.

For the case $r = 0$, $s = 1$, T_1 is given by $T_1 = \{(0; 2), (0; 3), (1; 0), (1; 1)\}$ and the following figure illustrates the selection of vertices into T_1.

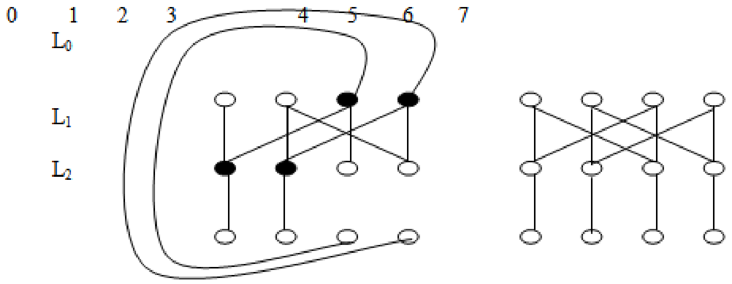


Fig. 2

The mirror image of T_1 in the right copy, denoted by T_2contains the vertices $\{(0; r_1), (0; r_2), (1; r_3), (1; r_4)\}$ where $r_1 + r + 2 = 7$, $r_2 + s + 2 = 7$, $r + r_3 = 7$, $s + r_4 = 7$. Clearly T_2 dominates all the vertices in the left copy of BF(3) \ F .

Then as per the above values of r and s, we get $T_2 = \{(0; 5), (0; 4), (1; 7), (1; 6)\}$.
The following figure illustrates the selection of vertices into T_2.

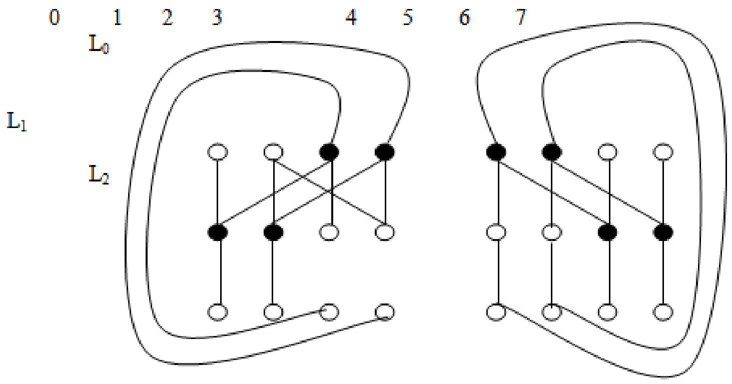


Fig. 7.10

Let $T = T_1 \cup T_2$. Then all vertices of BF(3) \ F are dominated by the eight selected vertices in T. and this domination is total domination. Therefore $\gamma_t(BF(3)) = \gamma_t(BF(3) \setminus F)$.

For other choices of edges in these two $K_{2,2}$ s, it can be shown that the total domination number of BF(3) and BF(3) \ F is unaltered. Similar is the case if $K_{2,2}$ s are taken between L_0 and L_1 in the right copy .

Similarly for any choice of five edges in BF(3) between L_1, L_2 and L_0, L_2, it can be shown that the total domination number of BF(3) and BF(3) \ F is unaltered.

Thus $b(BF(3)) = 4$. □

Lemma 3 : The total bondage number of BF(4) is 6.

Proof : Consider the graph BF(4). Then $\gamma_t(BF(4)) = 16$ (Lemma 5.3, Chapter 5 of [11]. Let T denote a total dominating set of BF(4).

By Recursive Construction 1, we know that BF(4) has two copies of BF(3) and a level L_3 with 2^4 vertices. From Lemma 7.7 , removal of five edges from BF(3), does not increase the total domination number of it. So select a total dominating set of BF(4) from two copies of BF(3).

Let (k; m) and (k+1; m′) be any two vertices in the left copy of BF(4). Suppose e is an edge incident on these two vertices .Let F denote the set of all edges incident on (k; m) and (k+1; m′) except ‘e’ so that $|F| = 6$. Then the two vertices (k; m) and (k+1; m′) dominate only each other and no other vertex of BF(4) \ F. Hence any total dominating set of BF(4) \ F must include these two vertices.

Now to dominate the remaining vertices of the left copy of BF(4) , include eight vertices into T. Similarly from the right copy , include eight vertices into T. Now these 16 vertices in T dominate all vertices of BF(4) \ F , except the vertices (k; m) and (k+1; m′) , since all edges incident on these vertices are deleted except ‘e ’ . Hence including these vertices into T, the cardinality of a total dominating set of BF(4) becomes 18 . This is the minimum cardinality , as any set of cardinality 16 can not total dominate BF(4) \ F . Thus γ_t(BF(4) \ S) > γ_t(BF(4)) = 16.

Hence b_t(BF(4)) = 6.

Now we claim that removal of any 5 edges from BF(4) does not increase the total domination number of BF(4). This is proved by taking the edges in $K_{2,2}$ s , between L_0 , L_1 and L_1 , L_2 .

Case 1 : Suppose F = {{ (0; r), (1; r)}, {(0; r), (3; r)}, {(0; r), (3; r + 1)}, {(0; s), (1; s)}, { (0; s), (3; s)}}, where r = 0,1 and $|r - s|$ = 1. Consider the graph BF(4) \ F. The selection of vertices for domination in BF(4) \ F is as follows. Let T_1 denote a total dominating set of BF(4) \ F.

Consider a $K_{2,2}$ between L_0 and L_1 incident on four vertices (0; r), (0; r + 2), (1; r), (1; r + 2) in the left copy where r = 0, 1. Since the edge incident on vertices (0; r) , (1; r) in the first $K_{2,2}$ is deleted, select the vertex (0; r + 2) or (1; r + 2) into T_1 . First select a vertex of L_1, say (1; r + 2) into T_1 . This vertex dominates two vertices (0; r + 2) , (0; r) of L_0 and one vertex (2; r + 2) of L_2 .

Consider another $K_{2,2}$ between L_0 and L_1 incident on four vertices (0; s), (0; s + 2), (1; s), (1; s + 2) in the left copy , where $|r - s|$ = 1. Again the edge incident on two vertices (0; s) , (1; s) is deleted , either the vertex (1; s + 2) or (0; s + 2) is to be selected into T_1 . Select the vertex (1; s + 2) into T_1 . This vertex dominates two vertices (0; s + 2) , (0; s) of L_0 and one vertex (2; s + 2) of L_2 .

Then T_1 = {(1; r + 2), (1; s + 2) } .

It is observed that there are four $K_{2,2}$ s between L_0 and L_1 in the left copy of BF(4) . As the next $K_{2,2}$ s are mirror image of the first $K_{2,2}$ s , select the mirror image of T_1 in these $K_{2,2}$ s . So T_1 becomes T_1 = {(1; r + 2), (1; s + 2) , (1; r_1), (1; s_1) } where $r_1 = r + 2^2$, $s_1 = s + 2^2$.

Here (1; r_1), (1; s_1) dominate the vertices **(**0; r_1), (0; r_1 + 2) and **(**0; s_1), (0; s_1 + 2) respectively. Thus all vertices of L_0 in the left copy are dominated.

Consider four $K_{2,2}$ s between L_1 and L_2 incident on vertices {(1; r_2), (1; $r_2 + 2^2$) , (2; r), (2 ; $r_2 + 2^2$) } , {(1; s_2), (1; $s_2 + 2^2$) , (2; s_2), (2 ; $s_2 + 2^2$) } , {(1; p), (1; p + 2^2) , (2; p), (2 ; p + 2^2) } , {(1; q), (1; q + 2^2) , (2; q), (2 ; q + 2^2) } respectively , where $r_2 \neq s_2 \neq p \neq q$ and r_2 = 0, 1, 2,….7 in the left copy of BF(4) \ F Then select the vertices (2; r_2), (2; s_2), (2; p), (2; q) of L_2 into T_1, as already the vertices of L_1 are selected into T_1 and any total dominating set of BF(4) \ F contains vertices from two consecutive levels .

Now T_1 = { (1; r + 2), (1; s + 2) , (1; r_1), (1; s_1) (2; r_2), (2; s_2), (2; p), (2; q) }.

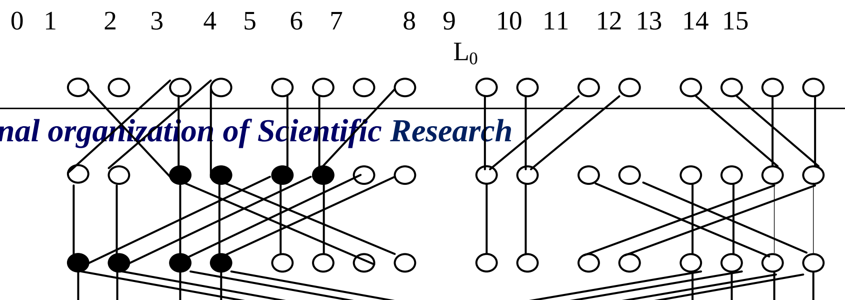
Here (2; r_2) dominates (1; r_2), (1; $r_2 + 2^2$) ; (2; s_2) dominates (1;s_2), (1; $s_2 + 2^2$) ; (2; p) dominates (1; p), (1; p + 2^2) ; (2; q) dominates (1; q), (1; q + 2^2) . Thus eight vertices of L_1 are dominated in the left copy of BF(4) \ F . Now the selected vertices in L_1 into T_1 viz., (1; r + 2), (1; s + 2), (1; r_1) = (1; r + 2^2), (1; s_1) = (1; s + 2^2) dominate respectively the vertices (2; r + 2 + 2^2), (2; s + 2 + 2^2), (2; r_1) = (2; r + 2^2) , (2; s_1) = (2; s + 2^2) of L_2 . Thus including the selected vertices of L_2 , all the vertices of L_2 in the left copy of BF(4) \ F are dominated .

The selected vertices in L_2 dominate the vertices (3; r_2), (3; s_2), (3; p), (3; q) of L_3 in the left copy of BF(4) \ F and also (3; $r_2 + 2^3$), (3; $s_2 + 2^3$), (3; p+ 2^3), (3; q + 2^3) of L_3 in the right copy of BF(4) \ F . Now the vertices (3; $r_2 + 2^2$), (3; $s_2 + 2^2$), (3; p + 2^2), (3; q + 2^2) of L_3 in the left copy of BF(4) \ F are undominated .

For the values of r = 0 , s = 1 , r_2 = 0 , s_2 = 1 , p = 2, q = 3 ,

T_1 = {(1; 2), (1; 3), (1; 4), (1; 5), (2; 0), (2; 1), (2; 2), (2; 3) }.

The following figure illustrates the selection of vertices into T_1 .



L_1

L_2

L_3

Fig. 3

From the figure , observe that the vertices in T_1, total dominate all the vertices except $(3; r_2 + 2^2) = (3; 4)$, $(3; s_2 + 2^2) = (3; 5)$, $(3; p + 2^2) = (3; 6)$, $(3;q+2^2) = (3;7)$ in the left copy of BF(4) \ F .

The mirror image of T_1 in the right copy, denoted by T_2 contains the vertices $\{(1; r^1), (1; s^1), (1; r_1^1), (1; s_1^1), (2; r_2^1), (2; s_2^1), (2; p^1), (2; q^1)\}$ where $r + 2 + r^1 = 15$, $s + 2 + s^1 = 15$, $r_1 + r_1^1 = 15$, $s_1 + s_1^1 = 15$, $r_2 + r_2^1 = 15$, $s_2 + s_2^1 = 15$, $p + p^1 = 15$, $q + q^1 = 15$. Then as per the above values of r , s , r_2, s_2,p,q, we get $T_2 = \{(1; 13), (1; 12), (1; 11), (1; 10), (2; 15), (2; 14), (2; 13), (2; 12)\}$.

As above it can be shown that the vertices in T_2 dominate all the vertices in the right copy of BF(4) \ F except the vertices $(3; r_2^1 - 7)$, $(3; s_2^1 - 5)$, $(3; p^1 - 3)$, $(3; q^1 - 1)$ of L_3 in the right copy of BF(4) \ F .

The following figure illustrates the selection of vertices into T_2 .

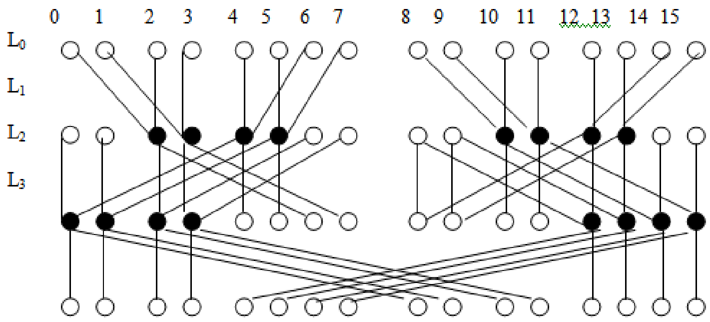


Fig.4

Again o bserve from the figure that T_2 dominates all the vertices in the right copy of BF(4) \ F except the vertices $(3; r_2^1 - 7) = (3; 8)$, $(3; s_2^1 - 5) = (3; 9)$, $(3; p^1 - 3) = (3; 10)$, $(3; q^1 - 1) = (3; 11)$ in the right copy of BF(4) \ F . Now the undominated vertices (3; 4), (3; 5), (3; 6), (3; 7) in the left copy are dominated by the selected vertices (2; 12), (2; 13), (2; 14), (2; 15) and the undominated vertices (3; 8), (3; 9) , (3; 10), (3; 11) in the right copy is dominated by the selected vertices (2; 0), (2; 1) , (2; 2), (2; 3) .

Let $T = T_1 \cup T_2$. Thus all vertices of BF(4) \ F are dominated by the 16 selected vertices in T. Therefore γ_t (BF(4)) = γ_t (BF(4) \ F) .

For other choices of edges in these two $K_{2,2}$ s, it can be shown that the domination number for BF(4) and BF(4) \ F is unaltered. Similar is the case if $K_{2,2}$ s are taken between L_0 , L_1 and L_1 , L_2 in the right copy .

Similarly for any choice of five edges in BF(4) between L_1 , L_2 and L_2 , L_3 or L_2 , L_3 and L_3 , L_0 , again it can be shown that the total domination number of BF(4) and BF(4) \ F is unaltered. Thus b(BF(4)) = 6. □

Lemma 4 : The total bondage number of BF(n) is 6.

Proof : From Recursive Construction 1, we know that for $n > 4$, every BF(n) has a copy of BF(4) between the first 4 levels L_0, L_1, L_2 and L_3. Removal of 6 edges incident on a pair of adjacent vertices from BF(4) increases the total domination number of this copy of BF(4). Since all these copies are disjoint this increases the total domination number of BF(n).

Thus b_t(BF(n)) = 6. □

The above results from Lemma 7.6 through Lemma 7.9 can be compiled as follows :

Theore 1 : The total bondage number of BF(n) is

$$b_t(BF(n)) = 3 \quad \text{for } n = 2$$
$$= 6 \quad \text{for } n \geq 3. \ \square$$

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