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# Triple Domination Number of a Butterfly Graph

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***Abstract:*** *Domination Theory is an important branch of Graph Theory that has wide range of applications to various branches of Science and Technology. A new family of graphs called Butterfly Graphs is introduced recently and study of its parameters is under progress. Butterfly Graphs are undirected graphs and are widely used in interconnection networks. A subset D of V(G) is called a triple dominating set of G if every vertex in V is dominated by at least three vertices in D. Cardinality of the minimum triple dominating set is called the triple domination number of G and is denoted by* $\gamma_{x3}(G)$ *.In this paper we present results about triple domination number of Butterfly Graphs BF(n) .*

***Keywords:*** *Butterfly graph, Domination set, Triple domination set*

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## I. MAIN RESULTS

**Lemma 1:** The Triple domination number of BF (2) is 6.

**Proof:** Consider butterfly graph BF (2). For a vertex to be triple dominated at least 3 vertices adjacent to that vertex are needed. Let D denote a triple dominating set of BF (2). Then D must have at least 3 vertices so that $|D| > 3$.

Consider a vertex v in a level, say $L_1$. To triple dominate this vertex, all its three adjacent vertices which are in $L_0$ are required. So these three vertices of $L_0$ are taken into D. Now one vertex, say u is left in $L_0$. Obviously the left over single vertex in $L_0$ does not dominate v and is not adjacent to the selected vertices in D, because they are the same level vertices. This vertex in $L_0$ is adjacent to the remaining 3 vertices of $L_1 \backslash v$. So include these 3 vertices into D. Thus the vertex u in $L_0$ and vertex v in $L_1$ are triple dominated by the vertices in D.

Since the graph is BF (2), every vertex in one level is adjacent to 3 vertices in the other level. Hence the vertices in D are also triple dominated. Therefore all the vertices of BF (2) are triple dominated by vertices of D and $|D| = 6$. Therefore $\gamma_{x3}(BF(2)) = 6$. □

**Note**: Observe that the selected vertices in D are consecutive vertices in $L_0$ and $L_1$. Otherwise a triple dominating set can not be obtained.

**Lemma .2:** The Triple domination number of BF (3) is 16.

**Proof:** Let $D_1$ denote a triple dominating set of BF (3). Consider a vertex $(1; m_1)$ in the left copy of BF (3). To triple dominate this vertex, include three adjacent vertices of $(1; m_1)$ into $D_1$. The vertex $(1; m_1)$ is adjacent to two vertices in $L_0$, viz., $(0; m_1)$, $(0; m_2)$, $m_1 \neq m_2$ and $|m_1 - m_2| = 2$ which belong to $K_{2,2}$ between $L_0$ and $L_1$ and two vertices in $L_2$ viz., $(2; m_1)$ , $(2; s_1)$ , $m_1 \neq s_1$ and $|m_1 - s_1| = 2^2$ which belong to $K_{2,2}$ between $L_1$ and $L_2$ .

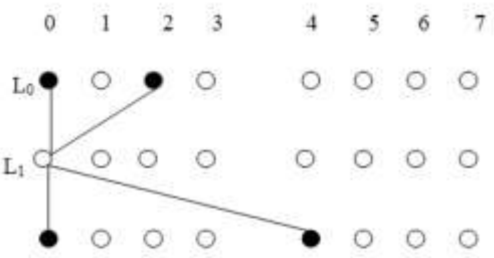
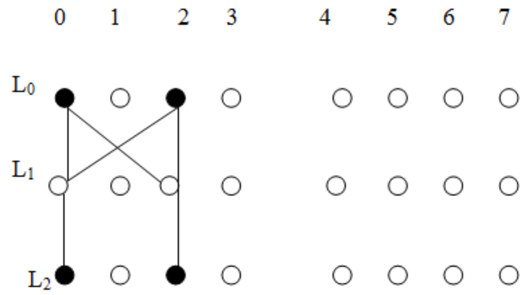



**Fig 1**

Without loss of generality, select two vertices from $L_0$ and one vertex from $L_1$ into $D_1$. Then $D_1 = \{(0; m_1), (0; m_2), (2; m_1)\}$.

Consider the vertex $(1; m_2)$ in the left copy. This vertex belongs to the same $K_{2,2}$ of $(1; m_2)$ between $L_0$ and $L_1$ and this vertex is also 2-dominated by $(0; m_1)$ and $(0; m_2)$. To triple dominate this vertex, include one of its adjacent vertices (2; $m_2$), $m_1 \neq m_2$ and $|m_1 - m_2| = 2$ of $L_2$ into $D_1$.

Then $D_1 = \{(0; m_1), (0; m_2), (2; m_1), (2; m_2)\}$.




Next consider (1; $t_1$) in the left copy and $m_1 \neq t_1 \neq m_2$ and $|m_1 - t_1| = 2^0$ and $|m_2 - t_1| = 2^0$. This vertex belongs to $K_{2,2}$ between $L_0$ and $L_1$ and is 2-dominated by the vertices (0; $t_1$) and (0; $t_2$) and is single dominated by the vertex (2; $t_1$). Hence select thes vertices into $D_1$ .

Now consider the vertex (1; $t_2$) in the left copy. This vertex belongs to $K_{2,2}$ of (1; $m_2$) between $L_0$ and $L_1$ and is 2- dominated by (0; $t_1$) and (0; $t_2$) and is single dominated by (2; $t_2$) . As(0;$t_1$)and (0; $t_2$) are already selected into $D_1$ , include (2; $t_2$) into $D_1$ .

So $D_1 = \{(0; m_1), (0; m_2), (0; t_1), (0; t_2), (2; m_1), (2; m_2), (2; t_1), (2; t_2)\}$.

Thus the vertices in $D_1$ triple dominate all the vertices in $L_1$ in the left copy. We now show that the vertices in $D_1$ are also triple dominated. Since the vertices in $D_1$ belong to $L_0$ and $L_2$ , the vertices in $L_0$ , 2-dominate the vertices of $L_2$ and vice-versa. Lastly these vertices are triple dominated by themselves.

Thus all vertices in the left copy are triple dominated.

As it was explained in Chapter 2 , the concept of mirror image, the right copy of BF(n) is a mirror image of left copy of BF(n) because of the symmetric structure of BF(n). Further this mirror image is also explained in terms of compliment of vertices.

The compliment set of $D_1 = \{(0; m_i), (2; m_i) / i = 0…3\}$ is given by $D_2 = \{(0; m_i), (2; m_i) / i = 4…7\}$. As $D_1$ , triple dominate all the vertices in the left copy , $D_2$ also triple dominate all the vertices in the right copy.

Hence $D = D_1 \cup D_2$ becomes a triple dominating set of BF(3) with $|D| = 2^4$ .

We claim that D is a minimum triple dominating set of BF(3). If possible let there exist a triple dominating set S such that $|S| < |D|$ .

By the symmetric structure of BF(n), the domination number must be even. Therefore $|S|$ is even. Suppose $|S| = 14$. Then by symmetry 7 vertices in the left copy and 7 vertices in the right copy are selected. But as the domination number is even this is not a choice. Let $|S| = 12$. Then 6 vertices in the left copy and 6 vertices in the right copy are selected. Since $\gamma_{x3}$ (BF(2)) = 6 , the selected 6 vertices in the left copy can not triple dominate 12 vertices. Similar is the case with the right copy . Therefore $|S| < |D|$ .

Hence D is a minimum triple dominating set of BF(3).

Thus $\gamma_{x3}(BF(3)) = 2^4$. □

**Possible choice of triple dominating sets in BF(3)**

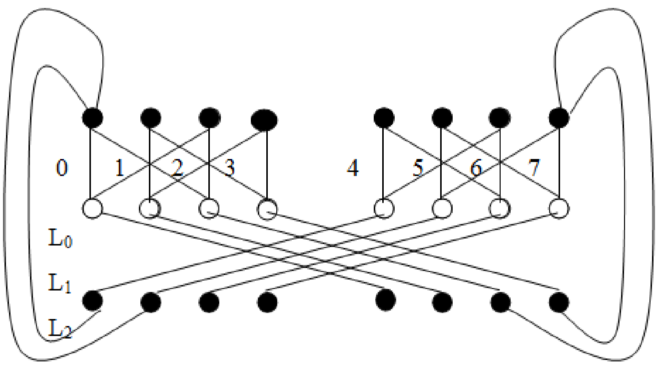



**Fig. 2**

**Theorem 3:** The triple domination number of BF (n) for n = 3k is 2k .$2^n$.

**Proof:** The theorem is proved by using the Principle of Mathematical Induction on k.

**Step 1:** Let k = 1. So n = 3. From Lemma 6.2, the result is true for BF (3). Hence the result is true for k = 1.

**Step 2 :** Let us assume that the result is true for k = t. We prove the result for k = t +1. Consider the graph BF(3(t+1)). From Recursive Construction 2, BF(3t+3) is decomposed into 8 copies of BF(3t) without wings and the last 3 levels form a pattern of BF(3) with $2^3$ vertex groups where each vertex group has $2^{3t}$ vertices. From the induction hypothesis the result is true for BF(3t).

So consider a triple dominating set $S_i$ of cardinality $2t.2^{3t}$ for BF(3t). In BF(3t+3) the last 3 levels form a pattern of BF(3), which is isomorphic to BF(3) without wings. From Lemma 6.2, the choice of vertices in a triple dominating set D for BF(3), is from $L_0$ and $L_2$ ( $L_0$ ,$L_1$ or $L_1$ , $L_2$ ) . Hence the winged edges together with straight, slant edges are used for triple domination. Therefore the result obtained for the graph BF(3) can be extended to a pattern of BF(3).

Thus a triple dominating set $D_3$ is obtained with $8.2.2^{3t}$ vertices for the last 3 levels, because there are 8 vertex groups such that each vertex group has $2.2^{3t}$ vertices.

Suppose a triple dominating set of cardinality less than $8.2.2^{3t}$ is obtained for the pattern of BF(3). Since the pattern of BF(3) is isomorphic to the graph BF(3) without wings, this assumption gives the existence of a triple dominating set of cardinality less than $2.2^3$ for BF(3), which is a contradiction as $\gamma_{\times 3}(BF(5)) = 2.2^3$.

Hence the cardinality of $D_3$ is $8.2.2^{3t}$ only .

Let $D = \bigcup_{i=1}^{8} S_i \cup D_3$ .

As $S_i$ are minimum triple dominating sets of disjoint copies of BF(3t) between levels $L_0$ to $L_{3t-1}$ and $D_3$ is a minimum triple dominating set for a pattern of BF(3) between levels $L_{3t}$ to $L_{3t+2}$ , it follows that D is a disjoint union of minimum triple dominating sets.

Hence D is a minimum triple dominating set of BF(3t+3) whose cardinality is

$$|D| = 8.2t.2^{3t} + 8.2.2^{3t} .$$
$$= 2(t+1)\,2^{3t+3} .$$
$$= 2(t+1)\,2^{3(t+1)} .$$

So the result is true for k = t + 1.

Hence by the Principle of Mathematical Induction the result is true for all positive integers k.

Thus $\gamma_{\times 3}(BF(3k)) = 2k.2^{3k}$. □

**One such selection of a triple dominating set for BF(6) is illustrated in the following figure**.

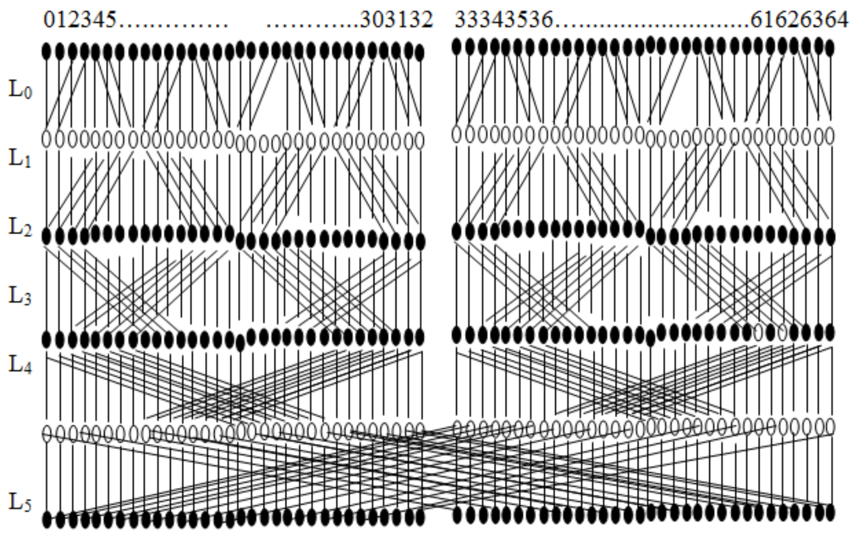



**Fig. 3**

**Corollary 4:** The Triple domination number of BF(3k+1) is $(2k+1).2^n$.

**Proof:** By Recursive Construction 1, there are 2 copies of BF(3k) and a level with $2^{3k+1}$ vertices in BF(3k+1). Let D denote a triple dominating set of BF(3k+1). Consider the vertices in a minimum triple dominating set for BF(3k) and include these vertices into D. The choice of vertices into a minimum triple dominating set for left and right copy of BF(3k) is from $L_0$ to $L_{3k-1}$ which triple dominate the vertices of $L_0$, $L_1$ and $L_{3k-1}$.

Now the winged edges between $L_0$ and $L_{3k-1}$ are deleted and extended to $L_{3k}$ . Hence $2^{3k+1}$ vertices of $L_0$ and $L_{3k-1}$ are single dominated. To triple dominate these $2^{3k+1}$ vertices of $L_0$ and $L_{3k-1}$ respectively, include $2^{3k+1}$ vertices of $L_{3k}$ into D. As a minimum triple dominating set is to be constructed , there is no other choice , other than including these vertices into D. Thus D becomes a minimum triple dominating set.

So $\gamma_{x\,3}(BF(3k+1)) = 2.\gamma_{x3}(BF(3k)) + 2^{3k+1}$ .
$= 2\,.2k.2^{3k} + 2^{3k+1}$ .
$= 2\,k\,.2^{3k+1} + 2^{3k+1}$ .
$= (2k+1)\,2^{3k+1}$ . □

**Two possible choices of triple dominating sets in BF (4)**

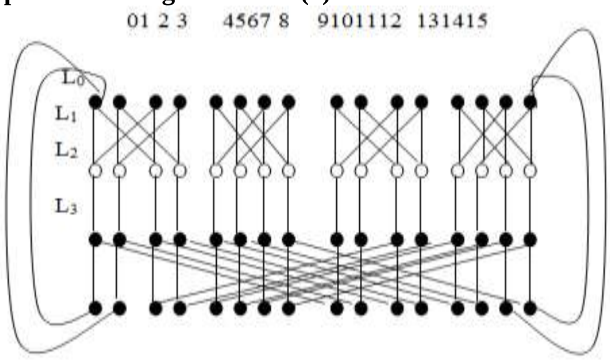



**Fig. 4**

**Corollary 5:** The triple domination number of BF(3k+2) is $(k+1)\,2^{3k+3}$.

**Proof:** Consider BF(3k+2). By the Recursive Construction 1, there are $2^2$ copies of BF(3k) and two levels $L_{3k}$, $L_{3k+1}$ each having $2^{3k+2}$ vertices in BF(3k+2). Let D denote a minimum triple dominating set of BF(3k+2). First include the vertices in a minimum triple dominating set of BF(3k) into D. The choice of vertices in a minimum triple dominating set for left and right copy of BF(3k) includes vertices from $L_0$ to $L_{3k-1}$ , which triple dominate the vertices of $L_0$ , $L_1$ and $L_{3k-1}$.Now the winged edges between $L_0$ and $L_{3k-1}$ are deleted and extended to $L_{3k+1}$ . Hence $2^{3k+2}$ vertices of $L_0$ and $L_{3k-1}$ are single dominated. To triple dominate these vertices , include $2^{3k+2}$ vertices of $L_{3k}$ and $2^{3k+2}$ vertices of $L_{3k+1}$ into D. Thus all the vertices of BF(3k+2) are triple dominated and D becomes a minimum triple dominating set of BF(3k+2).

Hence $\gamma_{x\,3}(BF(3k+2)) = 2^2\,.\,\gamma_{x\,3}(BF(3k)) + 2.2^{3k+2}$ .
$= 2^2\,.\,2\,k.2^{3k} + 2.2^{3k+2}$ .
$= 2(k+1)\,2^{3k+2}$ .
$= (k+1)\,2^{3k+3}$ . □

**Two such selections of a triple dominating sets for BF(3k+2) for k = 1 is illustrated in the foll owing figure.**

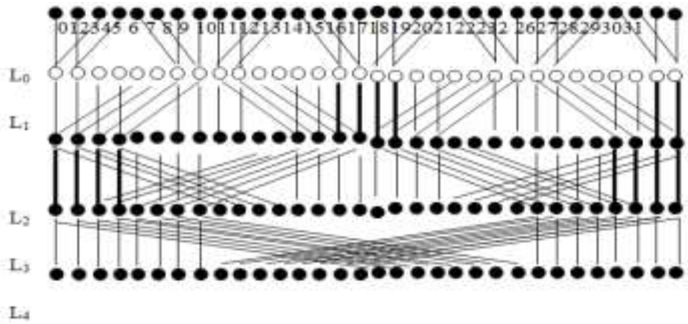



**Fig. 5**

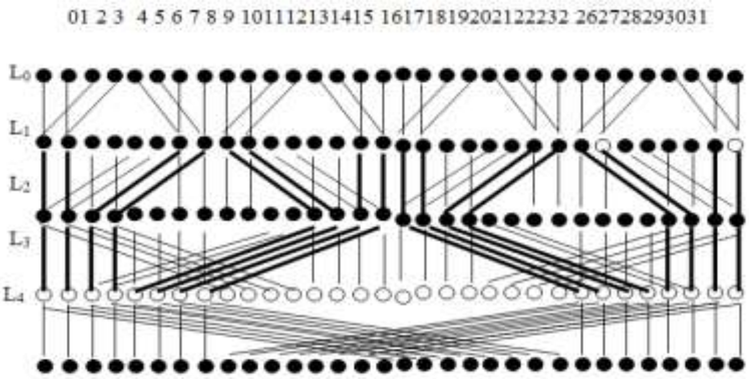



**Fig. 6**

The results through Lemma 6.1 to Corollary 6.5 can be compiled as,
**Theorem 6:** The triple domination number of BF(n) is

$$\gamma_{x\,3}(BF(n)) = \begin{cases} 6 & \text{if } n = 2 \\ 2.\,2^{3} & \text{if } n = 3 \\ 2\,.\,k\,.2^{3k} & \text{if } n = 3k \\ (2k+1)\,2^{3k+1} & \text{if } n = 3k+1 \\ (k+1)\,2^{3k+3} & \text{if } n = 3k+2. \end{cases}$$ □

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