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An Interpolation Process on Weighted (0,1,2;0)- Interpolation on Laguerre Polynomial

R. Srivastava[1] and Geeta Vishwakarma[2]
Department of Mathematics and Astronomy, Lucknow University, Lucknow , INDIA 226007
Corresponding Author: R. Srivastava

Abstract: The aim of this pape r is to study a special kind of mixed type of interpolation on the zeros of $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$ and on the interval [0,∞) with boundary (Hermite) conditions gives a simultaneous approximation to a differentiable function and its derivative under what conditions . We have investigated the existence, uniqueness, explicit representation and estimation of interpolatory polynomial .
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I. INTRODUCTION

Pál [6] proved that when the function values are prescribed on one set of n points and derivative values on other set of n-1 points , then there exist no unique polynomial of degree ≤ 2n-2 , but prescribing function value at one more point not belonging to former set of n points there exists a unique polynomial of degree ≤ 2n-1. . Pál [6] considered an interscaled set of nodes which were the zeros of some polynomial $P(x)$ and its derivative $P^{'}(x)$. The weighted lacunary interpolation was studied and modified by mathematicians such as Szili L., Mathur P. and Datta S. ,Mathur K.K. and Srivastava R. , Srivatava R and Mathur K.K.etc [9][4][5] [10]. Lénárd M. [3] investigated the Pál – type interpolation problem on the nodes of Laguerre abscissas.The aim of this paper is to study Pál – type interpolational polynomial with $\omega_{n+k}(x) = x^k L_n^{(k)}(x)$.we have determined the existence, uniqueness and explicit representation of fundamental polynomials of such special kind of mixed type of interpolation on interval [0,∞) . For this we have considered the problem in which $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ the two sets of interscaled nodal points.

(1.1) $$0 \le \xi_0 < \xi_1^* < \xi_1 < \cdots < \xi_{n-1} < \xi_n^* < \xi_n < \infty$$

on the interval [0,∞) . We seek to determine a polynomial $R_n(x)$ of minimal possible degree 4n+k satisfying the interpolatory conditions :

(1.2) $$R_n(\xi_i) = g_i, \qquad R_n^{'}(\xi_i) = g_i^*, \qquad (\rho R_n)^{''}(\xi_i) = g_i^{**}$$

$$R_n(\xi_i^*) = d_i^{***}, \quad = g_i^{***}, \qquad for\ i = 1(1)n$$

(1.3) $$R_n^{(j)}(\xi_0) = g_0^{(j)}, \qquad j = 0,1,\dots,k$$

where g_i , g_i^*, g_i^{**}, $g_i^{***}$and $g_0^{(j)}$ are arbitrary real numbers. Here Laguerre polynomials $L_n^{(k)}(x)$ and $L_n^{(k-1)}(x)$ have zeroes $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ respectively and $x_0 = 0$. We prove existence, uniqueness , explicit representation and estimation of fundamental polynomials with respect to the weight function $\rho(x) = e^{-x/2}x^{(k+1)/2}$.

II. PRELIMINARIES

In this section we shall give some well-known results which are as follws :
As we know that the Laguerre polynomial is a constant multiple of a confluent hypergeometric function so the differential equation is given by

(2.1) $$xD^2L_n^k(x) + (1 + k - x)DL_n^k(x) + nL_n^k(x) = 0$$

(2.2) $$L_n^{(k-1)^{'}}(x) = -L_{n-1}^{(k)}(x)$$

Also using the identities

(2.3) $L_n^{(k)}(x) = L_n^{(k+1)}(x) - L_{n-1}^{(k+1)}(x)$

(2.4) $xL_n^{(k)'}(x) = nL_n^{(k)}(x) - (n+k)L_{n-1}^{(k)}(x)$

We can easily find a relation

(2.5) $\frac{d}{dx}[x^k L_n^k(x)] = (n+k)x^{k-1}L_n^{(k-1)}(x)$

By the following conditions of orthogonality and normalization we define Laguerre polynomial $L_n^{(k)}(x)$, $for\ k > -1$

(2.6) $\int_0^\infty e^{-x}x^k L_n^{(k)}(x)L_m^{(k)}(x)dx = \Gamma(k+1)\binom{n+k}{n}\delta_{nm}$ $\qquad n,m = 0,1,2,\ldots..$

(2.7) $L_n^{(k)}(x) = \sum_{\mu=0}^{n}\binom{n+k}{n}\frac{(-x)^\mu}{\mu!}$

The fundamental polynomials of Lagrange interpolation are given by

(2.8) $l_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)'}(x_j)(x-x_j)} = \delta_{i,j}$

(2.9) $l_j^*(x) = \frac{L_n^{(k-1)}(x)}{L_n^{(k-1)'}(x_j)(x-x_j)} = \delta_{i,j}$

(2.10) $l_j^{*'}(y_j) = \begin{cases} \frac{L_n^{(k-1)'}(y_i)}{L_n^{(k-1)'}(y_j)(y_i-y_j)} & i \neq j \\ -\frac{(k-y_j)}{2y_j} & i = j \end{cases} \qquad i,j = 1(1)n$

(2.12) $l_j'(y_j) = \frac{1}{(y_j-x_j)}[\frac{L_n^{(k)'}(y_j)}{L_n^{(k)'}(x_j)} - \frac{L_n^{(k)}(y_j)}{L_n^{(k)'}(x_j)(y_j-x_j)}]$ $\qquad , \ j = 1(1)n$

For the roots of $L_n^{(k)}(x)$ we have

(2.13) $2\sqrt{x_j} = \frac{1}{\sqrt{n}}[j\pi + O(1)]$

(2.14) $\eta(x)|S_n^{(l)}(x)| = O(1)$, $\qquad$ where $\eta(x)$ is the weight function

(2.15) $\left|L_n^{(k)'}(x_j)\right| \sim j^{-k-\frac{3}{2}}n^{k+1}$, $\qquad (0 < x_j \leq \Omega, n = 1,2,3,\ldots\ldots.)$

(2.16) $|L_n^k(x)| = \begin{cases} x^{-\frac{k}{2}-\frac{1}{4}}\ O\left(n^{\frac{k}{2}-\frac{1}{4}}\right), & cn^{-1} \leq x \leq \Omega \\ O(n^k), & 0 \leq x \leq cn^{-1} \end{cases}$

(2.17) $O(l_j(x))= O\left(l_j^*(x)\right) = 1$,

III. NEW RESULT

Theorem 3.1 : For n and k fixed positive integer let $\{g_i\}_{i=1}^n$, $\{g_i^*\}_{i=1}^n$, $\{g_i^{**}\}_{i=1}^n$, $\{g_i^{***}\}_{i=1}^n$ and , $\{g_0^{(j)}\}_{j=0}^k$ are arbitrary real numbers then there exists a unique polynomial $R_n(x)$ of minimal possible degree ≤ 4n+k on the nodal points (1.1) satisfying the condition (1.2) and (1.3). The polynomial $R_n(x)$ can be written in the form

(3.1) $\boldsymbol{R_n(x) = \sum_{j=1}^{n} U_j(x)g_j + \sum_{j=1}^{n} V_j(x)g_j^* + \sum_{j=1}^{n} W_j(x)g_j^{**} + \sum_{j=1}^{n} X_j(x)g_j^{***} + \sum_{j=0}^{k} C_j(x)g_0^{(j)}}$

where $U_j(x)$, $V_j(x)$, $W_j(x)$, $X_j(x)$ and $C_j(x)$ are fundamental polynomials of degree ≤ 4n+k given by

(3.2) $U_j(x) = \frac{x^{k+1}\{l_j(x)\}^3 L_n^{k-1}(x)}{x_j^{(k+1)} L_n^{(k-1)}(x_j)}\left[1-(x-x_j)\left\{\frac{(1+k-5x_j)}{2x_j} + \tilde{c_j}(x-x_j)\right]\right.$

(3.3) $V_j(x) = \frac{x^{k+1}L_n^k(x)L_n^{(k-1)}(x)[l_j(x)]^2[1-2x_j(x-x_j)]}{x_j^{(k+1)}L_n^{(k-1)}(x_j)L_n^{(k)'}(x_j)}$

(3.4) $W_j(x) = \frac{e^{x_j/2}x^{k+1}\, l_j(x)\ [L_n^{(k)}(x)]^2\ L_n^{(k-1)}(x)}{x_j^{\frac{3}{2}(k+1)}\, L_n^{(k-1)}(x_j)\ [L_n^{(k)'}(y_j)]^2}$,

(3.5) $X_j(x) = \frac{x^{k+1}\ l_j^*(x)\ [L_n^{(k)}(x)]^3}{y_j^{k+1}\ [L_n^{(k)}(y_j)]^3}$,

(3.6) $C_j(x) = p_j(x)x^j\,[L_n^k(x)]^2[L_n^{(k-1)}(x)]^2$
$+x^k L_n^{(k-1)}(x)\ [L_n^{(k)}(x)]^2\left[c_j^* - \frac{L_n^{k-1}(x)p_j^*(x)+q_j^*(x)L_n^{(k)}(x)}{x^{k-j}}\right]$,
$j = 0,1,\dots,k-1$

(3.7) $C_k(x) = \frac{1}{k!L_n^{(k-1)}(0)[L_n^k(0)]^3}x^k L_n^{(k-1)}(x)[L_n^{(k)}(x)]^3$

where $p_j(x)$ and $q_j(x)$ are polynomials of degree at most (k-j-1) and c_j^* is an arbitrary constant.

Theorem 3.2 Let the interpolatory function $f:\mathcal{R}\to\mathcal{R}$ be continuously differentiable such that,

$C(m) = \{f(x): f \text{ is continuous in } [0,\infty), f(x) = O(x^m) \text{ as } x\to\infty;$

where $m \geq 0$ *is an integer* , then for every $f\in C(m)$ and $k\geq 0$

(3.8) $R_n(x) = \sum_{j=1}^n \alpha_j^{**}U_j(x) + \sum_{j=1}^n \beta_j^{**}V_j(x) + \sum_{j=1}^n \gamma_j^{**}W_j(x) + \sum_{j=0}^k \varphi_0^{**(j)}C_j(x)$

satisfies the relations

(3.9) $\rho(x)|R_n(x) - \mathrm{f(x)}| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right)\omega\left(f,\frac{\log n}{\sqrt{n}}\right)$, for $0\leq x\leq cn^{-1}$

(3.10) $\rho(x)|R_n(x) - \mathrm{f(x)}| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right)\omega\left(f,\frac{\log n}{\sqrt{n}}\right)$, for $cn^{-1}\leq x\leq\Omega$

where ω is the modulus of continuity.

IV. PROOF OF THEOREM

Let $U_j(x)$, $V_j(x)$, $W_j(x)$, $X_j(x)$ and $C_j(x)$ are polynomials of degree ≤ 4n+k satisfying conditions (4.1) , (4.2) , (4.3), (4.4) and (4.5) respectively.

(4.1)
$$\begin{cases} U_j(x_i) = \begin{cases}0 & for\ i\neq j\\ 1 & i=j\end{cases}, \quad U_j'(x_i) = 0, \quad [\rho(x)U_j(x)]''_{x=x_i} = 0 \\ and \\ U_j(y_i) = 0, \quad U_j^{(l)}(0) = 0, \end{cases}$$

$i = 1(1)n \ \ and \quad l = 0,1,\dots,k$

(4.2)
$$\begin{cases} V_j(x_i) = 0, \quad V_j'(x_i) = \begin{cases}0 & for\ i\neq j\\ 1 & i=j\end{cases}, \quad [\rho(x)V_j(x)]''_{x=x_i} = 0 \\ and \\ V_j(y_i) = 0, \quad V_j^{(l)}(0) = 0, \end{cases}$$

$i = 1(1)n \ \ and \quad l = 0,1,\dots,k$

$$(4.3)\qquad \begin{cases} W_j(x_i)=0\,, \quad W_j'(x_i)=0\,, \quad [\rho(x)W_j(x)]''_{x=x_i}=\begin{cases}0 & for\ i\neq j\\ 1 & i=j\end{cases} \\ and \\ W_j(y_i)=0\,, \quad W_j^{(l)}(0)=0\,, \end{cases}$$

$$i=1(1)n \;\; and \qquad l=0,1,\dots,k$$

$$(4.4)\qquad \begin{cases} X_j(x_i)=0\,, \quad X_j'(x_i)=0\,, \quad [\rho(x)X_j(x)]''_{x=x_i}=0\,, \\ and \\ X_j(y_i)=\begin{cases}0 & for\ i\neq j\\ 1 & i=j\end{cases}, \quad X_j^{(l)}(0)=0\,, \end{cases}$$

$$i=1(1)n \;\; and \qquad l=0,1,\dots,k$$

$$(4.5)\qquad \begin{cases} C_k(x_i)=0\,, \quad C_j'(x_i)=0\,, \quad [\rho(x)C_j(x)]''_{x=x_i}=0\,, \\ and \\ C_k(y_i)=0 \quad C_k^{(l)}(0)=\begin{cases}0 & for\ i\neq j\\ 1 & i=j\end{cases}, \quad i=1(1)n \end{cases}$$

$$i=1(1)n \;\; and \qquad l=0,1,\dots,k$$

To determine $X_j(x)$ let

(4.6) $\qquad X_j^*(x)=C_1\, x^{k+1} l_j^*(x)[L_n^{(k)}(x)]^3$

where C_1 is arbitrary constants . $l_j^*(x)$ is defined in (2.9) . $X_j^*(x)$ is a polynomial of degree ≤ 4n+k

By using (2.8) we determine

(4.7) $\qquad C_1 = \frac{1}{y_j^{k+1}\,[L_n^{(k)}(y_j)]^3}$

Hence we find the fundamental polynomial $X_j(x)$of degree ≤ 4n+k

To find fundamental polynomial $W_j(x)$, let

(4.8) $\qquad W_j^*(x)=C_2\, x^{k+1}[L_n^{(k)}(x)]^2 L_n^{(k-1)}(x) l_j(x)$

where C_2 is an arbitrary constants , $l_j(x)$ is defined in (2.8) .$W_j^*(x)$ is polynomial of degree ≤ 4n+k satisfying the conditions (4.3) by which we obtain

(4.9) $\qquad C_2=\frac{e^{x_j/2}}{x_j^{\frac{3}{2}(k+1)}\, L_n^{(k-1)}(x_j)\;[L_n^{(k)'}(x_j)]^2}$

Hence we find the fundamental polynomial $W_j(x)$ of degree ≤ 4n+k

To find fundamental polynomial $V_j(x)$ let

(4.10) $\;V_j^*(x)=C_3x^{k+1}[l_j(x)]^2L_n^{(k)}(x)L_n^{(k-1)}(x)+C_4\, l_j(x)[L_n^{(k)}(x)]^2L_n^{(k-1)}(x)$

using conditions (4.2) finding the value of all the constants we obtain the fundamental polynomial $V_j(x)$ of degree ≤ 4n+k.

Again let

(4.11) $\qquad U_j^*(x)=x^{k+1}[l_j(x)]^3L_n^{(k-1)}(x)C_5+C_6(x-x_j)]+C_7x^{k+1}l_j(x)L_n^{(k-1)}(x)[L_n^{(k)}(x)]^2$

where C_5 , C_6 and C_7 are arbitrary constants .$l_j(x)$ is defined in (2.8) .$U_j^*(x)$ is polynomial of degree ≤ 4n+k satisfying the conditions (4.1) by which we obtain the fundamental polynomial $U_j(x)$ of degree ≤ 4n+k.

To find $C_j(x)$, we assume $C_j(x)$ for fixed $j\,\epsilon\,\{0,1,\dots\dots.,k-1\}$

(4.13) $C_j{}^*(x) = p_j^*(x)x^j\,[L_n^k(x)]^2[L_n^{(k-1)}(x)]^2 + x^k L_n^{(k-1)}(x)[L_n^{(k)}(x)]^2 g_n^*(x)$

where $p_j^*(x)$ and $g_n^*(x)$ are polynomials of degree (k-j-1) and n respectively . Now it is clear that $C_j^{(l)}(0) = 0\; for\; (l = 0, \dots, j-1)$ and since $L_n^{(k)}(x_i) = 0$ and $L_n^{(k-1)}(y_i) = 0$ we get $C_j(x_i) = 0\; and\; C_j(y_i) = 0 \quad for\; i = 1(1)n$. The coefficient of the polynomial $p_j^*(x)$ are calculated by

(4.14) $$C_j^{(l)}(0) = \frac{d^l}{dx^l}\left[p_j^*(x)x^j\,[L_n^k(x)]^2[L_n^{(k-1)}(x)]^2\right]_{x=0} = \delta_{i,j}$$
$$(l = j, \dots\dots, k-1)$$

Now using the condition $[\rho(x)C_k^*(x)]''_{x=x_i} = 0\;\; of$, we get

(4.15) $$g_n^*(y_i) = -(x_i)^{j-k} L_n^{k-1}(x_i)p_j^*(x_i)$$

which implies $g_n^*(x)$ as follows

(4.16) $$g_n^*(x) = -\frac{L_n^{k-1}(x)p_j^*(x) + q_j^*(x)L_n^{(k)}(x)}{x^{k-j}}$$

Using (4.13) and (4.16) we obtain $C_j(x)$ of degree $\le$ 4n+k satisfying the conditions (4.5).

V. ESTIMATION OF THE FUNDAMENTAL POLYNOMIALS

Lemma 5.1 Let *the fundamental polynomial* $X_j(x)$, *for* $j = 1,2,\dots,n$ *be given by* (3.5), *then we have*

(5.1) $$\sum_{j=1}^{n} e^{y_j/2} y_j^{\,-(k+1)/2}\left|X_j(x)\right| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right), \qquad \text{for } 0 \le x \le cn^{-1}$$

(5.2) $$\sum_{j=1}^{n} e^{y_j/2} y_j^{\,-(k+1)/2}\left|X_j(x)\right| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right), \qquad \text{for } cn^{-1} \le x \le \Omega$$

where $X_j(x)$ is given in equqtion (3.5) .

Proof : From (3.5) we have

(5.3) $$\sum_{j=1}^{n} e^{y_j/2} y_j^{\,-(k+1)/2}\left|X_j(x)\right| \le \sum_{i=1}^{n} \frac{|x^{k+1}|\;\; l_j^*(x)\;\; [L_n^{(k)}(x)]^3}{\left|y_j^{k+1}\right| [L_n^{(k)}(y_j)]^3}$$

By equations (2.13), (2.16) and (2.17) we yield the result.

Lemma 5.2 Let *the fundamental polynomial* $W_j(x)$, *for* $j = 1,2,\dots,n$ *given by* ((3.4) , *then we have*

(5.4) $$\sum_{j=1}^{n}\left|W_j(x)\right| = O\left(n^{-\frac{k}{2}-\frac{5}{2}}\right), \qquad \text{for } 0 \le x \le cn^{-1}$$

(5.5) $$\sum_{j=1}^{n}\left|W_j(x)\right| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right), \qquad \text{for } cn^{-1} \le x \le \Omega$$

where $W_j(x)$ is given in equation (3.4)

Proof : From (3.4) we have

$$\left|W_j(x)\right| \le \frac{e^{x_j/2}|x^{k+1}|\, l_j(x)\;\; [L_n^{(k)}(x)]^2 \left|\, L_n^{(k-1)}(x)\right|}{\left|x_j^{\frac{3}{2}(k+1)}\right|\left|L_n^{(k-1)}(x_j)\right|\, [L_n^{(k)'}(y_j)]^2}$$

(5.6) $$\sum_{j=1}^{n}\left|W_j(x)\right| \le \sum_{j=1}^{n} \frac{e^{x_j/2}|x^{k+1}|\, l_j(x)\;\; [L_n^{(k)}(x)]^2 \left|\, L_n^{(k-1)}(x)\right|}{\left|x_j^{\frac{3}{2}(k+1)}\right|\left|L_n^{(k-1)}(x_j)\right|\, [L_n^{(k)'}(y_j)]^2}$$

Thus by using (2.13) and (2.17) , equations (5.4) and (5.5) follows.

Lemma 5.3 Let *the fundamental polynomial* $V_j(x)$, *for* $j = 1,2,\dots,n$ *be given by* (3.3) , *then we have*

(5.6) $$\sum_{j=1}^{n} e^{x_j/2} x_j^{\,-(k+1)/2}\left|V_j(x)\right| = O\left(n^{-\frac{k}{2}-\frac{3}{2}}\right), \qquad \text{for } 0 \le x \le cn^{-1}$$

(5.7) $$\sum_{j=1}^{n} e^{x_j/2} x_j^{\,-(k+1)/2}\left|V_j(x)\right| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right), \qquad \text{for } cn^{-1} \le x \le \Omega$$

where $V_j(x)$ is given in (3.3).

Proof : From (3.3) we have ,

$$\left|V_j(x)\right| \le \frac{|x^{k+1}|[l_j(x)]^2\left|L_n^{(k-1)}(x)\right|\left|L_n^k(x)\right|}{\left|x_j^{(k+1)}\right|\left|L_n^{(k-1)}(x_j)\right|\left|L_n^{(k)'}(x_j)\right|} + \frac{|x^{k+1}|\,|l_j(x)|\left|L_n^{(k-1)}(x)\right|\left|2x_j\right|[L_n^k(x)]^2}{\left|x_j^{(k+1)}\right|\left|L_n^{(k-1)}(x_j)\right|[L_n^{(k)'}(x_j)]^2}$$

(5.8) $\sum_{j=1}^{n} e^{x_j/2} x_j^{-(k+1)/2} |V_j(x)|$

$$\leq \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| [l_j(x)]^2 |L_n^{(k-1)}(x)| |L_n^{k}(x)|}{|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)| |L_n^{(k)'}(x_j)|} + \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| l_j(x) |L_n^{(k-1)}(x)| |2x_j| [L_n^{k}(x)]^2}{|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)| [L_n^{(k)'}(x_j)]^2}$$

$$= \zeta_{6,1} + \zeta_{6,2}$$

where

$$\zeta_{6,1} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| [l_j(x)]^2 |L_n^{(k-1)}(x)| |L_n^{k}(x)|}{|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)| |L_n^{(k)'}(x_j)|}$$

$$\zeta_{6,2} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| [l_j(x)]^2 |L_n^{(k-1)}(x)| |2x_j| [L_n^{k}(x)]^2}{|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)| [L_n^{(k)'}(x_j)]^2}$$

owing to (2.13) and (2.16), we get the result.

. **Lemma 5.4** Let *the fundamental polynomial* $U_j(x)$, *for* $j = 1,2,\dots,n$ *be given by* (3.2) , *then we have*

(5.9) $\sum_{j=1}^{n} e^{x_j/2} x_j^{-(k+1)/2} |U_j(x)| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right)$, for $0 \leq x \leq cn^{-1}$

(5.10) $\sum_{j=1}^{n} e^{x_j/2} x_j^{-(k+1)/2} |U_j(x)| = O\left(n^{-\frac{k}{2}-\frac{1}{2}}\right)$, for $cn^{-1} \leq x \leq \Omega$

where $U_j(x)$ is given in (3.2)

Proof : From (3.2) we have

$$|U_j(x)| \leq \frac{|x^{k+1}| |\{l_j(x)\}^3| |L_n^{k-1}(x)|}{|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|} + \frac{|x^{k+1}| \{|l_j(x)\}^3| |(x-x_j)| |L_n^{k-1}(x)| |1+k-5x_j|}{2|x_j| x_j^{(k+1)} L_n^{(k-1)}(x_j)} + \frac{|x^{k+1}| |\{l_j(x)\}^3| |L_n^{k}(x)| \left|\tilde{c_j}\,(x-x_j)^2\right|}{2\,|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|}$$

(5.11) $\sum_{j=1}^{n} e^{x_j/2} x_j^{-(k+1)/2} |U_j(x)|$

$$\leq \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| |\{l_j(x)\}^3| |L_n^{k-1}(x)|}{|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|} + \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| |\{l_j(x)\}^3| |(x-x_j) L_n^{k-1}(x)| |1+k-5x_j|}{2|x_j| |x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|} + \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| |\{l_j(x)\}^3| |L_n^{k}(x)| \left|\tilde{c_j}\,(x-x_j)^2\right|}{2\,|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|}$$

$$= \zeta_{6,1} + \zeta_{6,2} + \zeta_{6,3}$$

where

$$\zeta_{6,1} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| \{l_j(x)\}^3 |L_n^{k-1}(x)|}{|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|}$$

$$\zeta_{6,2} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| |\{l_j(x)\}^3| |(x-x_j) L_n^{k-1}(x)| |1+k-5x_j|}{2|x_j| |x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|}$$

$$\zeta_{6,3} = \sum_{j=1}^{n} \frac{e^{x_j/2} x_j^{-(k+1)/2} |x^{k+1}| |\{l_j(x)\}^3| |L_n^{k}(x)| \left|\tilde{c_j}\,(x-x_j)^2\right|}{2\,|x_j^{(k+1)}| |L_n^{(k-1)}(x_j)|}$$

Thus by using (2.13) and (2.17) , we yield the result.

Now we state our main theorem in § 6.

VI. PROOF OF MAIN THEOREM 3.2

Since $R_n(x)$ given by equation (6.1) is exact for all polynomial $Q_n(x)$ of degree $\leq$ 4n+k , we have

(6.1) $Q_n(x) = \sum_{j=1}^{n} Q_n(x_j) U_j(x) + \sum_{j=1}^{n} Q_n{}'(x_j) V_j(x) + \sum_{j=1}^{n} [\rho(x) Q_n(x)]''_{x=x_j} W_j(x) + \sum_{j=1}^{n} Q_n(y) X_j(x) + \sum_{j=0}^{k} Q_n(x_0) C_j(x)$

From equation (6.2.1) and (6.4.1) we get

(6.2) $\rho(x)|f(x) - R_n(x)| \leq \rho(x)|f(x) - Q_n(x)| + \rho(x)|Q_n(x) - R_n(x)|$

$$\leq \rho(x)|f(x) - Q_n(x)| + \sum_{j=1}^{n} \rho(x)\left|f(x_j) - Q_n(x_j)\right| \left|U_j(x)\right|$$

$$+\sum_{j=1}^{n} \rho(x)\left|f'(x_j) - Q_n{}'(x_j)\right| \left|V_j(x)\right| + \sum_{j=1}^{n} [\rho(x)Q_n(x)]''_{x=x_j} \left|W_j(x)\right|$$

$$+\sum_{j=1}^{n} \rho(x)\left|f(y_j) - Q_n(y_j)\right| \left|X_j(x)\right|$$

$$+\sum_{j=0}^{k} \rho(x)\left|f^l(x_0) - Q_n{}^l(x_0)\right| \left|C_j(x)\right|$$

Thus (6.2) ,and Lemmas 5.1 ,5.2, 5.3 and 5.4 completes the proof of the theorem.

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