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Topological Invariance of Separation Properties in Bitopological Spaces

*Harjot Singh
Assistant Professor, Department of Mathematics, Sikh National College, Banga.

Abstract. The main aim of this research study is to list some well known separation properties of bitopological spaces from available literature and to investigate and accomplish that how these separation properties are preserved under pair homeomorphism.

Keywords: Bitopological spaces, pair continuous, pair homeomorphism, separation properties.

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I. INTRODUCTION AND PRELIMINARIES

As far as the development of bitopological spaces are concerned, in 1963 Kelly (Kelly, 1963) gave the idea of bitopological spaces. Then this new concept of bitopological spaces is used by Kelly (Kelly, 1963) to study non-symmetric functions that introduce two arbitrary topologies on X. Further, in the same piece of research work, new concepts of pairwise Hausdorff, pairwise regular and pairwise normal, corresponding to the idea of separation axioms of topological space, are introduced in bitopological spaces and thoroughly investigated. Patty (Patty, 1967) Pervin (Pervin, 1967), Kim (Kim, 1968), Fletcher (Fletcher et al., 1969), Saegrove (Saegrove, 1971) and many other mathematicians carried out further research in the field of compactness, connectedness, total disconnectedness and more detailed separation properties in bitopological spaces.

In present research study, main focus is on listing of separation properties of bitopological spaces and to examine which of these separation properties are preserved under pair homeomorphism.

If τ_1 , τ_2 are arbitrary topologies on X then triplet (X, τ_1 , τ_2) is said to be a bitopological space on X. For any subset A of (X, τ_1, τ_2), τ_1 –cl(A) and τ_2 –cl(A) denote closure of A with respect to τ_1 and τ_2 respectively. Further, τ_1–open (τ_1–closed) and τ_2 –open (τ_2–closed) will be used to denote open (closed) set in a bitopological space (X, τ_1, τ_2) with respect to τ_1 and τ_2 respectively.

Definition 1 (Fletcher et al., 1969). (X, τ_1, τ_2) is said to be pairwise T_0 if and only if for each pair of distinct points x and y of X, there is either a τ_1-open set containing x but not y or there exists a τ_2 –open set containing y but not x.

Definition 2 (Reilly, 1970). (X, τ_1, τ_2) is said to be pairwise T_1 if and only if for each pair of distinct points x, y, there exists a τ_1-open set U and a τ_2-open set V such that $x\in U$, $y\notin U$ and $y\in V$, $x\notin V$.

Definition 3 (Kelly, 1963). (X, τ_1, τ_2) is said to be pairwise T_2 if and only if for each pair of distinct points x, y, there exists a τ_1-open set U and a τ_2-open set V such that $x\in U$, $y\in V$ and $U\cap V=\phi$.

Definition 4 (Kelly, 1963). In a bitopological space (X, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if for each point $x\in$ X and for each τ_1 -closed set A such that $x\notin A$, there exists a τ_1-open set U and a τ_2-open set V such that $x\in U$, $A \subseteq V$ with $U\cap V=\phi$.

Definition 5 (Kelly, 1963). Bitopological space (X, τ_1, τ_2) is said to be pairwise regular if τ_1 is regular with respect to τ_2 and τ_2 is regular with respect to τ_1.

Definition 6 (Saegrove, 1971). Bitopological space (X, τ_1, τ_2) is said to be pairwise T_3 if and only if it is pairwise regular and pairwise T_1.

Definition 7 (Saegrove, 1971). A function f from (X, τ_1 , τ_2) into (Y, τ'_1 , τ'_2) is pair continuous if and only if the induced functions f from (X, τ_1) into (Y, τ'_1) and (X, τ_2) into (Y, τ'_2) are pair continuous.

Definition 8 (Saegrove, 1971). A bitopological space (X, τ_1, τ_2) is pairwise completely regular if and only if for each τ_1-closed set A and for each point $x \notin A$, there exists a pair continuous function
$f : (X, \tau_1, \tau_2) \rightarrow ([0, 1], R, L)$ such that $f(x)=1$ and $f(A)=\{0\}$, and for each τ_2-closed set B and for each point $y \notin B$, there exists a pair continuous function $g : (X, \tau_1, \tau_2) \rightarrow ([0, 1], R, L)$ such that $g(y)=0$ and $g(B)=\{1\}$.

Definition 9 (Saegrove, 1971). Bitopological space (X, τ_1, τ_2) is said to be pairwise $T_{3\frac{1}{2}}$ if and only if it is pairwise completely regular and pairwise T_1.

Definition 10 (Kelly, 1963). Bitopological space (X, τ_1, τ_2) is said to be pairwise normal if and only if for each τ_1-closed set A and τ_2-closed set B disjoint from A, there exists a τ_1-open set U and a τ_2-open set V such that $A \subseteq V$, $B \subseteq U$ and $U \cap V = \phi$.

Definition11 (Saegrove, 1971). Bitopological space (X, τ_1, τ_2) is said to be pairwise T_4 if and only if it is pairwise pairwise normal and pairwise T_1.

Definition 12 (Saegrove, 1971). A function f from (X, τ_1, τ_2) into (Y, τ'_1, τ'_2) is said to be pair homeomorphic if and only if the induced functions f from (X, τ_1) to (Y, τ'_1) and (X, τ_2) to (Y, τ'_2) are pair homeomorphic.

II. INVARIANCE OF SEPARATION PROPERTIES UNDER PAIR HOMEOMORPHISM

Theorem 1. The property of being pairwise T_0 in a bitopological space is preserved under pair homeomorphism.
Proof. Suppose that (X, τ_1, τ_2) is a pairwise T_0 bitopological space and
$f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$ is a pair homeomorphism. Let x and y are any two distinct members of Y, there exist two different members x' and y' such that $f(x')=x$ and $f(y')=y$. Since, (X, τ_1, τ_2) is pairwise T_0, therefore there exists a τ_1-open set U such that $x' \in U$ but $y' \notin U$ or there exists a τ_2-open set V such that $y' \in V$ but $x' \notin V$. Evidently, $f(x') \in f(U)$ but $f(y') \notin f(U)$ or $f(y') \in f(V)$ but $f(x') \notin f(V)$. Map f is pair open, therefore f(U) is τ'_1-open set and f(V) is τ'_2-open set. Thus, any two distinct members x and y in Y there exists a τ'_1-open set f(U) such that $x=f(x') \in f(U)$ but $y=f(y') \notin f(U)$ or there exists a τ'_2-open set f(V) such that $y=f(y') \in f(V)$ but $x=f(x') \notin f(V)$. This completes the proof.

Theorem 2. The property of being pairwise T_1 in a bitopological space is preserved under pair homeomorphism.

Proof. Let (X, τ_1, τ_2) is a pairwise T_1 bitopological space and $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$ is a pair homeomorphism. Let x and y are any two distinct members of Y, there exists two different members x' and y' such that $f(x')=x$ and $f(y')=y$. Since, (X, τ_1, τ_2) is pairwise T_1, therefore there exists a τ_1-open set U and a τ_2-open set V such that $x' \in U$, $y' \notin U$ and $y' \in V$, $x' \notin V$. Clearly, $f(x') \in f(U)$, $f(y') \notin f(U)$ and $f(y') \in f(V)$, $f(x') \notin f(V)$. As f is pair open, therefore f(U) is τ'_1-open set and f(V) is τ'_2-open set. Thus, for any two distinct members x and y in Y there exists a τ'_1-open set f(U) and a τ'_2-open set f(V) such that $x=f(x') \in f(U)$, $y=f(y') \notin f(U)$ and $y=f(y') \in f(V)$, $x=f(x') \notin f(V)$. Hence, (Y, τ'_1, τ'_2) is pairwise T_1.

Theorem 3. The property of being pairwise T_2 in a bitopological space is preserved under pair homeomorphism.

Proof. Consider a pairwise T_2 bitopological space (X, τ_1, τ_2). Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$ is a pair homeomorphism. To prove required result, let x and y are any two distinct members of Y, therefore there exists two different members x' and y' such that $f(x')=x$ and $f(y')=y$. Since, (X, τ_1, τ_2) is pairwise T_2, therefore there exists a τ_1-open set U and a τ_2-open set V such that $x' \in U$, $y' \in V$ and $U \cap V = \phi$. It is obvious that $f(x') \in f(U)$, $f(y') \in f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \phi$. As f is pair open, therefore f(U) is τ'_1-open set and f(V) is τ'_2-open set. Thus, for any two distinct members x and y in Y there exists a τ'_1-open set f(U) and a τ'_2-open set f(V) such that $x=f(x') \in f(U)$, $y=f(y') \in f(V)$ and $f(U) \cap f(V) = \phi$. From this desired result follows.
Theorem 4. The property of being pairwise regular in a bitopological space is preserved under pair homeomorphism.

Proof. Let $f : (X, \tau_1, \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$ is a pair homeomorphism, where (X, τ_1, τ_2) is a pairwise regular bitopological space. To show that (Y, τ'_1, τ'_2) is also pairwise regular. Let y is any member of Y and A is any τ'_1-closed set such that $y \notin A$. Then, there exists x in X such that $y=f(x)$ also $x \notin f^{-1}(A)$, a τ_1-closed set. Since, (X, τ_1, τ_2) is pairwise regular, therefore there exists a τ_1-open set U and a τ_2-open set V such that $x \in U$, $f^{-1}(A) \subseteq V$ and $U \cap V = \phi$. It is obvious that $y=f(x) \in f(U)$, $A \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \phi$. As f is pair open, therefore f(U) is τ'_1-open set and f(V) is τ'_2-open set. Thus, for any τ'_1-closed set A not containing arbitrary $y \in Y$, there exists a τ'_1-open set f(U) and a τ'_2-open set f(V) such that $y=f(x)\in f(U)$, $A\subseteq f(V)$ and $f(U)\cap f(V)=\phi$.This proves that τ_1 is regular with respect to τ_2. Similarly, it can be proved that τ_2 is regular with respect to τ_1. This completes required proof.

Remark 1. By using above theorem it can be readily proved that pair homeomorphic image of pairwise T_3 bitopological space is also pairwise T_3.

Theorem 5. Any pair homeomorphic image of a pairwise completely regular bitopological space is pairwise completely regular.

Proof. Suppose that $f : (X, \tau_1 , \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$ is a pair homeomorphism, where (X, τ_1 , τ_2) is a pairwise completely regular bitopological space. To demonstrate that (Y, τ'_1, τ'_2) is also pairwise completely regular. Let y is any member of Y and A is any τ'_1-closed set such that $y\notin A$. Then, there exists x in X such that y=f(x) also $x\notin f^{-1}(A)$, a τ_1-closed set. Since, (X, τ_1, τ_2) is pairwise completely regular , therefore there exists a pair continuous function g: $(X, \tau_1 , \tau_2) \rightarrow ([0, 1], R, L)$ such that g(x)=1 and $g(f^{-1}(A))=\{0\}$. It means $g(f^{-1}(y))=(gof^{-1})(y)=1$ and $(gof^{-1})(A)=\{0\}$. As f is pair homeomorphism, therefore $f^{-1}: (Y, \tau'_1, \tau'_2) \rightarrow (Y, \tau_1 , \tau_2)$ is pair continuous and hence $gof^{-1}: (X, \tau'_1, \tau'_2) \rightarrow ([0, 1], R, L)$ is also pair continuous. Similarly, desired result can be attained for any member y' of Y and any τ'_2-closed set B is such that $y'\notin B$. Hence, (Y, τ'_1, τ'_2) is pairwise completely regular

Remark 2. By using above theorem it can be readily proved that pair homeomorphic image of pairwise $T_{3\frac{1}{2}}$ bitopological space is also pairwise $T_{3\frac{1}{2}}$.

Theorem 6. The property of being pairwise normal in a bitopological space is preserved under pair homeomorphism.

Proof. Consider a pair homeomorphism f: $(X, \tau_1 , \tau_2) \rightarrow (Y, \tau'_1, \tau'_2)$, here (X, τ_1 , τ_2) is a pairwise normal bitopological space. To prove that (Y, τ'_1, τ'_2) is also pairwise normal. Let A is any τ'_1-closed set and B is τ'_2-closed set such that $A\cap B=\phi$. Then, $f^{-1}(A)$ is a τ_1-closed set and $f^{-1}(B)$ is a τ_2-closed set such that $f^{-1}(A)\cap f^{-1}(B)=f^{-1}(A\cap B)=\phi$. Since, (X, τ_1 , τ_2) is pairwise normal, therefore there exists a τ_1-open set U and a τ_2-open set V such that $f^{-1}(B)\subseteq U$, $f^{-1}(A) \subseteq V$ and $U\cap V=\phi$. It is evident that $B\subseteq f(U)$, $A\subseteq f(V)$ and $f(U)\cap f(V)=f(U\cap V)=\phi$. As f is pair open, therefore f(U) is τ'_1-open set and f(V) is τ'_2-open set. Thus, (Y, τ'_1, τ'_2) is pairwise normal.

Remark 3. By using above theorem it can be easily accomplished that pair homeomorphic image of pairwise T_4 bitopological space is also pairwise T_4.

III. CONCLUSION

In this research work, it is established that separation properties pairwise T_0, pairwise T_1, pairwise T_2, pairwise regular, pairwise completely regular, pairwise $T_{3\frac{1}{2}}$, pairwise normal and paiwise T_4 are preserved under pair homeomorphism. In fact, pairwise T_0, pairwise T_1, pairwise T_2 are preserved under pair one-one, pair onto and pair open map.

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