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Coincidence Point and Common Fixed Point In *B*-Metricspaces

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Abstract: In this Paper, we proved coincidence and common fixed point in b- metric spaces. The presented theorems extend, generalize and improve result in the literature. Also, we introduce some definitions and examples the support the validity of our result.

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I. INTRODUCTION

*F*ixed point theory is one of the most important topics in the development of mathematical analysis. In this area of the simplest and most useful results in fixed point theory is Banach fixed point theorem. The first important and significant result was proved in 1922 for contraction mapping in complete metric space.

*L*et(X, d)be complete metric space and T be self mapping of$X$ satisfying
$(1\,)\,d(T_x, T_y\,) \leq kd(x, y)$ for all $x, y \in X$ where$k \in [0,1)$. Then T has unique fixed point.

A mapping satisfying the conditions(1) is called contraction mapping. In [3] Bakhin introduced b-mectric spaces as a generalized of metric spaces. He proved the contraction mapping principle in$b$- metric spaces as generalization the famous Banach contraction principle.
In this paper, we present some new fixed-point coincidence point and common fixed point in b- metric space. Throughout this paper $IRandIR^+$ will represent the set of real numbers and non – negative real number respectively

2 Preliminaries

Definition 1.[8]. Let X be a non- empty set and let d:$X \times X \to \mathbb{R}^+$ be a function satisfying the conditions.
$d_1)$ $d(x, y) = 0 \Leftrightarrow x = y$
$d_2)$ $d(x, y) = d(y, x)$;
$d_3)$ $d(x, y) \leq d(x, z) + \; d(z, y)$for all $x, y, z \in X$. Then d it is called metric on X, and the pair (X, d) is called metricspace.
Definition 2.[9]. Let X be a non-empty set. Let $k \geq 1$ be a real number then a mapping b:$X \times X \to \mathbb{R}^+$ is called b – metric if $\forall x, y, z \in X$, the following conditions are satisfied:
$b_1)$ $b(x, y) = 0 \;\Leftrightarrow x = y$
$b_2)$ $b(x, y) = b(y, x)$
$b_3)$ $b(x, y) \leq k[b(x, z) + b(z, y)]$.And the pair $(X, b)$is called b – metric space.
It is clear from the definition of b −metric that every metric space is b −metric for $k = 1$, but the converse is not true as clear from the following example.
Example 2.1. Le$X = \{0,1,2\}$. Defined $b: X \times X \to \mathbb{R}^+$ as follows
$b(0,0) = b(1,1) = b(2,2) = 0, b(1,2) = b(2,1) = b(0,1) = b(1,0) = 1, b(2,0) = b(0,2) = m \geq 2$ for $k = \frac{m}{2}$ where $m \geq 2$
the function defined above is a b-metric space but not a metric for $m > 2$.
Definition 3: A sequence $\{x_n\}$ in b-metric space (X, b) is called Cauchy sequence if for $\in > 0$ there exists a positive integer N such that for $m, n \geq N$ we have $b\left(x_{m,}x_n\right) < \in$
Definition 4: A sequence $\{x_n\}$ is called convergent in b-metric space (X, b) if for$\in > 0$ there exists for$n \geq N$we have $b\left(x_{n,}x\right) < \in$ where x is called the limit point of the sequence $\{x_n\}$.
Definition 5: A b-metric space (X, b) is said to be complete if every Cauchy sequence in X converge to a point of X
Lemma 1. Let (X, b) be a b-metric space and $\{x_n\}$ be a sequence in b-metric space such that

(3) $b(x_n, x_{n+1}) \leq \alpha . b(x_{n-1}, x_n)$
for n =1,2, 3,…. and $0 \leq \alpha k < 1, \alpha \in [0,1)$, and k is defined in b-metric space then $\{x_n\}$ is a Cauchy sequence in X.
Coincidence point: Given two mappings $f, g: X \to Y$, we say that a point x in X is coincidence point of f and g. If $f(x) = g(x)$. We can write $X = Y$ and we can take g is the identity mapping.
Commuting mapping: Two mapping $f, g: X \to X$ is said to commuting if

$$f(g(x)) = g(f(x)) \forall x$$
$$(fog)(x) = (gof)(x) \forall x$$

II. MAIN RESULTS

Theorem.3.1: Let (X, b) be a b-metric space and $f, g: Y \to X$, $f(Y) \subseteq g(Y)$ are mappings such that
(i) $b(f_x, f_y) \leq \alpha b(g_x, g_x) + \beta b(g_x, f_x) + \gamma b(f_y, g_y) + \mu[b(g_x, f_y) + b(g_y, f_x)] \forall x, y \in y$ where $\alpha, \beta, \gamma\mu \geq 0$ with $k\alpha + k\beta + \gamma + (k^2 + k)\mu < 1$
(ii) Either $f(Y)$ or $g(Y)$ is complete.
Then f, g have a coincidence point.

Proof: Let x_0, x_1 be points of Y such that $x_0 = gx_1$. Since $f(Y) \subseteq g(Y)$ and $fx_0 = gx_1$ hence we can contract a sequence $\{x_n\}$ s.t $fx_n = gx_{n+1}$. Let $z_n = fx_n$
Put $x = x_n y = x_{n+1}$ in (i)

$$b(fx_n, gx_{n+1}) = b(z_n, z_{n+1})$$
$$\leq \alpha b(gx_n, gx_{n+1}) + \beta(gx_n, fx_n) + \gamma b(fx_{n+1}, gx_{n+1})$$
$$+ \mu[b(gx_n, fx_{n+1}) + b(gx_{n+1}, fx_n)]$$
$$\leq \alpha b(z_{n-1}, z_n) + \beta b(z_{n-1}, z_n) + \gamma b(z_{n+1}, z_n) + \mu[b(z_{n-1}, z_{n+1}) + b(z_n, z_n)]$$
$$\leq \alpha b(z_{n-1}, z_n) + \beta b(z_{n-1}, z_n) + \gamma b(z_{n+1}, z_n) + \mu k[b(z_{n-1}, z_n) + b(z_n, z_{n+1})]$$
$$b(z_n, z_{n+1}) \leq \frac{\alpha + \beta + k\mu}{1 - (\gamma + k\mu)} b(z_{n-1}, z_n)$$
$$\leq \lambda b(z_{n-1}, z_n)$$

where $\lambda = \frac{\alpha+\beta+k\mu}{1-(\gamma+k\mu)}$ From conditions $\lambda = \frac{1}{k}$
so, from lemma (1) $z_n = \{fx_n\}$ is Cauchy sequence. Suppose $g(Y)$ is complete. Then $\exists p \in g(Y)$ s.t $z_n \to p$ and $\exists z \in y$ s.t g_z, p putting $x = x_n$ and $y = z$ in (1)
$b(fx_n, f_Z) \leq \alpha b(gx_n, g_Z) + \beta b(gx_n, fx_n) + \gamma b(f_Z, g_Z) + \mu[b(gx_n, f_Z) + b(g_Z, fx_n)]$
Taking limit $n \to \infty$
$b(p, f_Z) \leq \alpha b(p,p) + \beta b(p,p) + \gamma b(f_Z, g_Z) + \mu[b(p, f_Z) + b(p,p)]$
The inequality is possible only if $b(p, f_Z) = 0$
So $p = f(z) = g(z)$
Hence p is coincidence point of f *and* g.
Again if $f(Y)$ is complete then
$z_n \to p \in f(Y) \subset g(Y)$. Hence as above z is coincidence point of f and g.

In the fowling theorems and corollaries, we take $f(X) \subseteq g(X)$

Theorem3.2: Let (X, b) be a b −metric space with coefficient $k \geq 1$ and $f, g : X \to X$ are mappings such that

$$\text{(i) } b \leq \alpha(g_x, g_y) + \beta b(g_x, f_x) + \gamma(g_y, f_y) + \mu[b(g_x, f_y) + b(g_y, f_x)]$$

$\forall x, y \in X$ where $\alpha, \beta, \gamma, \mu \geq 0$ with $k\alpha + k\beta + \gamma + (k^2 + k)\mu < 1$
(ii) Either $f(X)$ *or* $g(X)$ is complete
(iii) f and g are commuting at their coincidence point.
Then f and g have unique common fixed point.

Proof: If we take $Y = X$ in theorem (3)then we get $z_n = fx_n$ such that $\{z_n\}$ is a Cauchy sequence. Suppose $g(X)$ is complete. Then $z_n \to p \in g(X)$ hence $\exists z \in X$ such that $g(z) = p$
Putting $x = x_n y = z$ in (3)
$f(z) = g(z) = p$. Since f and g are commuting at their coincidence point hence $fgz = gfz$ such that

$$f(p) = gp.$$

Now putting
$x = z, y = f(z)$ in (3)

$$b(f_Z, ff_Z) \leq \alpha b(g_Z, gf_Z) + \beta(g_Z, f_Z) + \gamma(gf_Z, ff_Z) + \mu[b(g_Z, ff_Z) + b(gf_Z, f_Z)]$$

$$b(p,f_p) \leq (p,g_p) + \beta b(p,p) + \gamma(g_p,f_p) + \mu[b(p,f_p) + b(g_p,p)]$$
$$b(p,f(p)) \leq 0$$

which is possible only if $p = f(p)$ So p $= f(p) = g(p)$. hence p is a common fixed point of f *and* g.
For unique, suppose p *and* p' are two common fixed point of f *and* g then from *(3)*

$b(f_p, f_{p'}) \leq \alpha b(g_p, g_{p'}) + \beta b(g_p, f_p) + \gamma b(g_{p'}, f_{p'}) + \mu[b(g_p, f_{p'}) + b(g_{p'}, f_p)]$

$$b(p,p') \leq \alpha b(p,p') + \beta b(p,p') + \gamma(p,p') + \mu[b(p,p') + b(p,p')]$$

$b(p,p') \leq 0$ *so* $p = p'$

Theorem 3.3 [6] Let (X, b) be a complete b-metric space with coefficient k ≥ 1 and f be a self-mapping $f: X \to X$ satisfying the condition

$$b(f_x, f_y) \leq \alpha. b(x,y) + \beta. b(x,f_x) + \gamma. b(y,f_y) + \mu. [b(x,f_y) + b(y,f_x)]$$

$\forall x, y \in X$, *where* $\alpha, \beta, \gamma, \mu \geq 0$, with $k\alpha + k\beta + \gamma + (k^2 + k)\mu < 1$ then f has a unique fixed point.
Proof: In the above theorem 3.2 if we take $g = I$ (identity mapping) then the theorem 3.3 automatically follows.

Corollary 3.1: Let (X, b) be a complete $b-$ metric space with coefficient $k \geq 1$ and $f, g: Y \to X$ be mappings satisfying the condition
$b(f_x, f_y) \leq \alpha b(g_x, g_y) + \beta b(g_x, f_x) + \gamma b(g_y, f_y) \forall x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ with $k\alpha + k\beta + \gamma < 1$, then f, g have coincidence point.
Proof: Putting $\mu = 0$ in theorem 3.1 we get the required result.

Corollary 3.2 Let (X, b) be a b $-$metric space with coefficient $k \geq 1$ and $f, g : X \to X$ are mappings such that

i. $b(f_x, f_y) \leq \alpha b(g_x, g_y) + \beta b(g_x, f_x) + \gamma b(g_y, f_y) \forall x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ with $k\alpha + k\beta + \gamma < 1$, then f, g have coincidence point.
ii. Either $f(X)$ *or* $g(X)$ is complete
iii. f and g are commuting at their coincidence point.

Then f and g have unique common fixed point.
Proof: Putting $\mu = 0$ in theorem 3.2 we get the required result.

Corollary 3.3 [6]: Let (X, b) be a complete b-metric space with coefficient $k \geq 1$ and a self-mapping $f: X \to X$ satisfying the condition

$$b(f_x, f_y) \leq \alpha. b(x,y) + \beta. b(x,f_x) + \gamma. b(y,f_y)$$

Where $\alpha, \beta, \gamma \geq 0$ with $k\alpha + k\beta + \gamma < 1$, then f has a unique fixed point.
Proof: In the above Corollary 3.2 if we take $g = I$ (identity mapping) then the Corollary 3.3 automatically follows.

Corollary3.4: Let (X, b) be a complete $b-$ metric space with coefficient $k \geq 1$ and f, g be mappings $f, g: Y \to X$ satisfying condition.
$b(f_x, f_y) \leq \alpha b(g_x, g_y) + \beta b(g_x, f_x) \forall x, y \in X$ where $\forall \alpha, \beta \geq 0$ with $k\alpha + k\beta < 1$.Then
f, g have coincidence point.

Proof: By putting $\mu = \gamma = 0$ in Theorem 3.1 we get the require result.

Corollary3.5 Let (X, b) be a b $-$metric space with coefficient $k \geq 1$ and $f, g : X \to X$ are mappings such that

i. $b(f_x, f_y) \leq \alpha b(g_x, g_y) + \beta b(g_x, f_x)) \forall x, y \in X$ where $\alpha, \beta, \geq 0$ with $k\alpha + k\beta < 1$,

ii. Either $f(X)$ *or* $g(X)$ is complete
iii. f and g are commuting at their coincidence point.

Then f and g have unique common fixed point.
Proof: By putting $\mu = \gamma = 0$ theorem 3.2 we get the required result.

Corollary 3.6[6] Let(X, b) be a complete b-metric space with coefficient $k \geq 1$ and f be a self mapping $f: X \to X$ satisfying the condition

$$b(f_x, f_y) \leq \alpha. b(x,y) + \beta. b(x,f_x)$$

∀ x, y ∈ X, where $\alpha, \beta \geq 0$ with $k\alpha + k\beta < 1$ then f has a unique fixed point.
Proof: In the above Corollary 3.5 if we take $g = I$ (identity mapping) then the Corollary 3.6 automatically follows.

Corollary3.7: Let (X, b) be a complete $b-$ metric space with coefficient $k \geq 1$ and $f, g: Y \rightarrow X$ be mappings satisfying the condition.
$b(f_x, f_y) \leq \alpha b(g_x, g_y) \forall x, y \in X$ where$\alpha \geq 0$ with $k\alpha < 1,$ $t$hen f, g coincidence point.
Proof: By putting $\beta = \gamma = \mu = 0$ in Theorem 3.1 we get the required result.

Corollary3.8 Let (X, b) be a b $-$metric space with coefficient$k \geq 1$and$f, g : X \rightarrow X$ are mappings such that
i. $b\left(f_x, f_y\right) \leq \alpha b\left(g_x, g_y\right) \forall x, y \in X$where $\alpha, , \geq 0$ with$k\alpha < 1,$
ii. Either$f(X)$ *or* $g(X)$is complete
iii. f and g are commuting at their coincidence point.
Then$f$ and g have unique common fixed point.
Proof: By putting $\beta = \gamma = \mu = 0$ itheorem 3.2 we get the required result.

Corollary3.9 [6] Let(X, b) be a complete b-space with coefficient $k \geq 1$ and f be a self mapping $f: X \rightarrow X$ satisfying the condition $b(f_x, f_y) \leq \alpha . b(x, y) \forall x, y \in X,$
where $\alpha \geq 0 k\alpha < 1$ then $f$has a unique fixed point.
Prof: In the above Corollary 3.8 if we take $g = I$ (identity mapping) then the Corollary 3.9 automatically follows.

III. CONCLUSION

If we take $X = Y$ and $g = I$ identity map in our result then result of “SARWAR” [6] is automatically follow from our result which is more generalized than the result of “SARWAR” [6].

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