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Characteristic Properties Of Subclasses For The Meromorphically Multivalent Functions

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Abstract: In the present paper, we introduce and investigate some new classes of multivalent functions involving linear operator and derive useful characteristic properties for Meromorphically multivalent functions. Several results are presented exhibiting relevant connections to some other results proved here and those obtained in earlier works.

Keywords: Analytic functions, Meromorphic functions, multivalent functions, linear operator and Hyper geometric function.

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I. INTRODUCTION

For any integer $m > -p$, let $\mathcal{T}_{p,m}$ denote the class of all meromorphic functions $f(z)$ of the form

$f(z) = z^{-p} + \sum_{r=m}^{\infty} a_r z^r \ (p \in \mathbb{N})$ (1)

which are analytic and p-valent in the punctured unit disk $\mathbb{D}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{D}/\{0\}$.

Now we define the general integral operator $\chi_p^n(\alpha,\beta)f(z)$ which is as follows

$$\chi_p^n(\alpha,\beta)f(z) = \frac{\beta}{\alpha} z^{-p-\left(\frac{\beta}{\alpha}\right)} \int_0^z t^{\left(\frac{\beta}{\alpha}+p-1\right)} \chi_p^{n-1}(\alpha,\beta)f(t)dt$$

$= \chi_p^1(\alpha,\beta)\left(\frac{1}{z^p(1-z)}\right) * \chi_p^1(\alpha,\beta)\left(\frac{1}{z^p(1-z)}\right) * \ldots * \chi_p^1(\alpha,\beta)\left(\frac{1}{z^p(1-z)}\right) \in f(z), z \in \mathbb{D}^*$ and $p \in \mathbb{N}$ (2)For the sake of convenience, in particular cases

(i) If $n = 0$ then the above integral operator $\chi_p^n(\alpha,\beta)f(z)$ converts into $\chi_p^0(\alpha,\beta)f(z) = f(z)$.

(ii) If $n = 1$ then the above integral operator $\chi_p^n(\alpha,\beta)f(z)$ converts into $\chi_p^1(\alpha,\beta)f(z) = \left(\frac{\beta}{\alpha}\right) z^{-p-\left(\frac{\beta}{\alpha}\right)} \int_0^z t^{\left(\frac{\beta}{\alpha}+p-1\right)} f(t)dt$.

(iii) If $n = 2$ then the above integral operator $\chi_p^n(\alpha,\beta)f(z)$ converts into $\chi_p^2(\alpha,\beta)f(z) = \left(\frac{\beta}{\alpha}\right) z^{-p-\left(\frac{\beta}{\alpha}\right)} \int_0^z t^{\left(\frac{\beta}{\alpha}+p-1\right)} \chi_p^1(\alpha,\beta) f(t)dt$.

Hence, if $f(z) \in \mathcal{T}_{p,m}$, we obtained the following results

$$\chi_p^n(\alpha,\beta)f(z) = \frac{1}{z^p} + \sum_{r=m}^{\infty} \left[\frac{\beta}{\beta+\alpha(r+p)}\right]^n a_r z^r \quad (3)$$

Therefore from (3), it is easy to see that

$$\alpha z\left(\chi_p^{n+1}(\alpha,\beta)f(z)\right)' = \beta\chi_p^n(\alpha,\beta)f(z) - (\alpha p+\beta)\chi_p^{n+1}(\alpha,\beta)f(z) \qquad (\alpha > 0) \quad (4)$$

We observe that (i) $\chi_p^\lambda(1,1)f(z) = \mathrm{P}_p^\lambda f(z)$ (see, [4]), (ii) $\chi_1^\lambda(1,j)f(z) = \mathrm{P}_j^\lambda f(z)$

We also see that

(i) $\chi_p^n(1,\beta)f(z) = \chi_{p,\beta}^n f(z)$ where $\chi_{p,\beta}^n f(z) = \frac{1}{z^p} + \sum_{r=m}^{\infty} \left[\frac{\beta}{\beta+(r+p)}\right]^n a_r z^r$

(ii) $\chi_p^n(\alpha,1)f(z) = \chi_{p,\alpha}^n f(z)$ where $\chi_{p,\alpha}^n f(z) = \frac{1}{z^p} + \sum_{r=m}^{\infty} \left[\frac{1}{1+\alpha(r+p)}\right]^n a_r z^r$

(iii) $\chi_p^n(1,1)f(z) = \chi_p^n f(z)$ where $\chi_p^n f(z) = \frac{1}{z^p} + \sum_{r=m}^{\infty} \left[\frac{1}{1+(r+p)}\right]^n a_r z^r$

II. DEFINITIONS

Let $\boldsymbol{\mathcal{T}}_{\boldsymbol{p,m}}^{\boldsymbol{n+1}}(\eta,\delta,\mu,\lambda)$ be the class of functions $f(z) \in \boldsymbol{\mathcal{T}}_{p,m}$ which is satisfies the condition

$$\Re e\left\{(1-\lambda)\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu} + \lambda\frac{\chi_p^n(\alpha,\beta)f(z)}{\chi_p^n(\alpha,\beta)g(z)}\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu-1}\right\} > \eta \quad (5)$$

where $g(z) \in \boldsymbol{\mathcal{T}}_p$ fulfill the following condition $\Re e\left\{\frac{\chi_p^{n+1}(\alpha,\beta)g(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right\} > \delta, \delta \in [0,1)$ (6)

Here μ *and* η belongs to real numbers such that $0 \le \eta < 1, \mu > 0$ and $\lambda \in c$ with $\Re e(\lambda) > 0$.

Currently many scholars and researchers studied similar classes for various types of different operators including Aouf and Mostafa [2], Al-Ashwah [6], [7] et al.

III. PRELIMINARY LEMMAS

The following lemmas will be required in our present investigation

Lemma i. Let Ω be a set in the complex plane c and let the function $\Psi: c^2 \to c$ satisfy the condition $\Psi(ir_2, s_1) \notin \Omega$ for allreal $r_2, s_1 \le -\frac{1+r_1^2}{2}$. If $q(z)$ is analytic in $with q(0) = 1$ and $\Psi(q(z), zq'(z)) \in \Omega, z \in \mathbb{D}$, then $\Re\{q(z)\} > 0 \ (z \in \mathbb{D})$.

Lemma ii. Let $q(z)$ is analytic function in disk $\mathbb{D}$ with $q(0) = 1$ and if $\alpha \in c/\{0\}$ with $\Re\{\alpha\} \ge 0$, then $\Re e\{q(z) + \alpha zq'(z)\} \ge \lambda, \quad \lambda \in [0,1) \Longrightarrow \Re e\{q(z)\} > \lambda + (1+\lambda)(2\gamma - 1)$ where γ is given by $\gamma = \gamma\,\{\Re e(\alpha)\} = \int_0^1 \{1 + t^{\Re e\{\alpha\}}\}^{-1} dt$ which isa function of $\Re e\{\alpha\}$ *and* $\frac{1}{2} \le \gamma < 1$.The estimate sharp in the sense that bound cannot be improved.

For the complex or a real number a,b and c (c$\neq$ 0and not negative integer) the Gauss hyper geometric function is defined by

$2^{F_1}(a, b; c; z) = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \cdots$

The above series converges absolutely for $z \in \mathbb{D}$ and $2^{F_1}(a, b; c; z)$ is an analytic function in the unit disk $\mathbb{D}$. Each of the identities is well known cf.; e.g., [7].

Lemma iii. For a real or complex parameters a, b and c (c $\neq$ 0)

$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} 2^{F_1}(a, b; c; z), \Re(c) > R(b) > 0$ (7)

$(1-tz)^{-a} 2^{F_1}\left(a, c-b; c; \frac{z}{z-1}\right) = 2^{F_1}(a, b; c; z)$ (8)

$2\ln 2 = 2^{F_1}\left(1,1; 2; \frac{1}{2}\right)$
(9)

IV. MAIN RESULTS

If it is not mentioned in other sense, let us assume that $(\alpha, \beta) > 0, p \in N, n \in \mathbb{N}_0$ *and* $-1 \le B < A \le 1$.

Theorem i. let $f(z) \in \boldsymbol{\mathcal{T}}_{p,m}(\eta, \delta, \mu, \lambda)$ then $\Re e\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu} > \left(\frac{2\mu\eta\beta + \alpha\beta\delta}{2\mu\beta + \alpha\beta\delta}\right), g(z) \in \boldsymbol{\mathcal{T}}_{p,m}(\eta, \delta, \mu, \lambda)$
(10)

Proof:let $= \frac{2\mu\eta\beta + \alpha\beta\delta}{2\mu\beta + \alpha\beta\delta}$, and we define the function $q(z)$ given by

$q(z) = \frac{1}{(1-\gamma)}\left\{\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right) - \gamma\right\}$ (11)

Then $q(z)$ is analytic in $\mathbb{D}$ and $q(0) = 1$.

If we consider the function $h(z)$ defined by $h(z) = \left(\frac{\chi^{n+1}(\alpha,\beta)g(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right)$ (12)

then by the our assumption (8), $\Re e\{h(z)\} > \delta$.

Differentiating (12) with respect to z and by the (6), we obtain $\frac{(1-\gamma)\alpha z q(z)'}{\beta\,\mu} = \left(\chi_p^{n+1}(\alpha,\beta)g(z) * \chi_p^{n}(\alpha,\beta)f(z) - \right.$
χpn+1α,βfz∗χpnα,βgz∗χpn+1α,βfzχpnα,βgz (13)

now using (13), we have

$$\{(1-\gamma)q(z) + \gamma\} + \frac{\alpha\lambda(1-\gamma)zq(z)'h(z)}{\mu\beta}$$

$$= \lambda\left(\frac{\chi_p^n(\alpha,\beta)f(z)}{\chi_p^n(\alpha,\beta)g(z)} - \frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right) + \left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu}$$

$$\Longrightarrow \{(1-\gamma)q(z) + \gamma\} + \frac{\alpha\lambda(1-\gamma)zq(z)'h(z)}{\mu\beta}$$

$= (1-\lambda)\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu} + \lambda\Gamma\left(\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right)\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu-1}$ (14)

Again, we consider the function $\Psi(r,s) = \left\{r - (r-1)\gamma + \frac{\alpha\lambda(1-\gamma)}{\beta\mu} Sh(z)\right\}$ (15)

From (15) and the fact that $f(z) \in \boldsymbol{\mathcal{T}}_{p,m}(\eta, \delta, \mu, \lambda)$, we have $\{\Psi\, q(z) zq'(z);\ z \in \mathbb{D}\} \subset \Omega$

$\Rightarrow \{w \in c: \Re e(w) > \eta\}$. Now for real $r_2, s_1 \leq -\frac{1+r_2^2}{2}$, we have

$$\Re e\{\Psi(ir_2, s_1)\}$$

$= \frac{\Re eh(z)(1-\gamma)\alpha\lambda s_1}{\mu\beta} + \gamma \leq -\frac{\delta(1+r_2^2)\alpha\lambda(1-\gamma)}{2\mu\beta} \leq -\frac{\lambda(1-\gamma)\delta}{2\mu\beta} = \eta$.Therefore $\forall z \in \mathbb{D}, \{\Psi(r_2, s_1)\} \notin \Omega$.

Thus from the lemma [1], we have $\Re eq(z) > 0 \ (z \in \mathbb{D})$ and hence

$$\Re e\left\{\frac{\chi^{n+1}(\alpha,\beta)f(z)}{\chi^{n+1}(\alpha,\beta)g(z)}\right\}^{\mu} > \gamma \ (z \in \mathbb{D})$$

Theorem ii. Let the function $f(z) and\ g(z)$ be in the class $\boldsymbol{\mathcal{T}}_{p,m}$ and let the function $g(z)$ satisfy the condition (7).If $\alpha \geq 1$ and $\Re e\left\{(1-\lambda)\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)} + \alpha\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right\} > \eta$Where $\eta \in [0,1)$and $(z \in \mathbb{D})$.

$$\Re e\left(\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right) > \gamma = \frac{(2\beta+\alpha\delta)+\alpha\lambda(\lambda-1)}{2\beta+\alpha\beta\lambda}$$

Proof: If we set$\lambda\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right\} = (\lambda-1)\left\{\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right\} + (1-\lambda)\left\{\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right\} + \lambda\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right\}$

By the theorem i and ii we have, $\lambda\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right\} \geq (\lambda-1)\left(\frac{2\eta\beta+\alpha\lambda\delta}{2\beta+\alpha\beta\delta}\right) + \eta$

$$\Rightarrow \lambda\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right\} \geq \left\{\frac{\alpha\delta(\lambda-1)+\eta(\lambda\delta+2\beta)}{2\beta+\alpha\delta\lambda}\right\}, \mu = 1 \text{ and } \alpha \geq 1, (z \in \mathbb{D})$$

Corollary i : Let $\in C/\{0\}\ with\ \Re e(\lambda) \geq 0$satisfies the following condition If $f(z) \in \boldsymbol{\mathcal{T}}_{p,m}$

$$\boldsymbol{\Re e}\left[\{(1-\lambda)(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\boldsymbol{\mu}} + \lambda(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z))\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\mu-1}\right] > \eta$$

Then $\Re e\left\{\left(z^p\chi_p^{n+1}(\alpha,\beta)f(z)\right)\right\}^{\mu} > \left(\frac{\alpha\Re e(\lambda)+2\mu\beta\eta}{\alpha\Re e(\lambda)+2\mu\beta}\right)$Further if $\lambda \geq 1\ and\ f(z) \in \boldsymbol{\mathcal{T}}_{p,m}$,satisfies

$\boldsymbol{\Re e}\left[(1-\lambda)\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z) + \lambda(\mathrm{z^p}\chi_p^{n}(\alpha,\beta)f(z)\right] > \eta.$

Then $\Re e\{\mathrm{z^p}\chi_p^{n}(\alpha,\beta)f(z)\} > \left(\frac{(2\beta\eta+\alpha\eta)+\alpha(\lambda-1)}{2\beta+\alpha\lambda}\right)$

Proof: If we taking $g(z) = \frac{1}{z^p}$ in theorem i and ii with $\Re e(\delta) = 1$, then the result is obvious.

Theorem iii. Let $\lambda \in C\ with\ \Re e(\lambda) \geq 0$. If $f(z) \in \boldsymbol{\mathcal{T}}_{p,m}$ satisfies the following condition:

$$\boldsymbol{\Re e}\left[(1-\lambda)\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\boldsymbol{\mu}} + \lambda(\mathrm{z^p}\chi_p^{n}(\alpha,\beta)f(z))\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\mu-1}\right] > \eta$$

Then$\boldsymbol{\Re e}\left[\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\boldsymbol{\mu}}\right] > \eta + (\mathbf{1}-\boldsymbol{\eta})(\mathbf{2}\boldsymbol{\rho}-\mathbf{1}),\ \rho = \frac{1}{2}F_2^1(1{,}1;\frac{\mu\beta}{\alpha\Re e(\lambda)}+1;\frac{1}{2})$

Proof: let $q(z) = \left(\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\boldsymbol{\mu}}\right)$ (16)

Then $q(z)$ is analytic with the condition $q(0) = 1$.Differentiating (16) with respect to z and using the identity (6), we get$q'(z)\frac{\alpha z}{\mu\beta} = \mathrm{z^p}\{\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\mu-1}\mathrm{z^p}\{\chi_p^{n}(\alpha,\beta)f(z)\} - \{\chi_p^{n+1}(\alpha,\beta)f(z)\}$

Now, by the definition of $q(z)$,we have$q(z) + zq'(z)\frac{\alpha z}{\mu\beta}$

$= \{\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\boldsymbol{\mu}} + \boldsymbol{\alpha}\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}^{\boldsymbol{\mu-1}}\{\mathrm{z^p}\chi_p^{n}(\alpha,\beta)f(z)\} - \{\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}$ Solving above mention equation, we get$\boldsymbol{q(z)} + \boldsymbol{zq'(z)}\frac{\boldsymbol{\alpha z}}{\boldsymbol{\mu\beta}}$

$$= (\mathbf{1}-\boldsymbol{\lambda})\{\mathbf{z^p}\boldsymbol{\chi_p^{n+1}(\alpha,\beta)f(z)}\}^{\boldsymbol{\mu}} + \boldsymbol{\lambda}\{\mathbf{z^p}\boldsymbol{\chi_p^{n}(\alpha,\beta)f(z)}\}\{(\mathbf{z^p}\boldsymbol{\chi_p^{n+1}(\alpha,\beta)f(z)}\}^{\boldsymbol{\mu-1}}$$

Therefore , by the assumption of theorem iii, we have$\Re e\left\{\boldsymbol{q(z)} + \boldsymbol{zq'(z)}\frac{\boldsymbol{\alpha z}}{\boldsymbol{\mu\beta}}\right\} > \eta$

By the lemma ii, $\Re e\{q(z)\} > \eta + (1-\eta)(2\rho-1)$Where $\rho = \rho(\Re e\{\lambda\} = \int_0^1\left(1 + t^{\alpha}\frac{\Re e(\lambda)}{\beta\mu}\right)^{-1}dt = \frac{\mu\beta}{\alpha\Re e(\lambda)}\int_0^1(1+\mu)^{-1}\mu^{\frac{\mu\beta}{\alpha\Re e(\lambda)}-1}d\mu$

Taking $\Re e(\lambda) = \lambda_1$ and$\lambda_1 > 0$. Therefore $\rho = \rho(\Re e\{\lambda\} = \int_0^1\left(1 + t^{\alpha}\frac{\lambda_1}{\beta\mu}\right)^{-1}dt$

Again using lemma iii, we have $\rho = \frac{1}{2}F_2^1(1{,}1;\frac{\mu\beta}{\alpha\lambda_1}+1;\frac{1}{2}$

Corollary ii: let $\lambda \in$ Real number with $\lambda \geq 1$. If $f(z) \in \boldsymbol{\mathcal{T}}_{p,m}$ satisfies $\Re e\left[(1-\lambda)\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\} + \right.$ λ(zp*χpnα,βfz>η*Then

$\Re e\left[\{\mathrm{z^p}\chi_p^{n}(\alpha,\beta)f(z)\}\right] > \eta + (1-\eta)(2\rho_1-1)\left(1-\frac{1}{\alpha}\right)$where$\rho_1 = \frac{1}{2}F_1^2(1{,}1;\frac{\beta}{\alpha\lambda}+1;\frac{1}{2})$.

Proof : let us consider $\lambda\Re e\{\mathrm{z^p}\chi_p^{n}(\alpha,\beta)f(z)\}$

$$= \Re e\left[(1-\lambda)\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\} + \lambda(\mathrm{z^p}\chi_p^{n}(\alpha,\beta)f(z) + (\lambda-1)\{(\mathrm{z^p}\chi_p^{n+1}(\alpha,\beta)f(z)\}\right]$$

By the theorem iii and above corollary $\Re e\left[z^{p}\chi_p^n(\alpha,\beta)f(z)\right]=\eta+(1-\eta)(2\rho_1-1)\left(1-\frac{1}{\alpha}\right)$where $\rho_1=\frac{1}{2}F_1^2(1,1;\frac{\beta}{\alpha\lambda}+1;\frac{1}{2})$.

Theorem iv. Let $f(z)and\ g(z)$ belongs to the class $\mathcal{T}_{p,m}$ and $g(z)$ satisfies the condition

$$\Re e\left\{(1-\lambda)\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu}+\lambda\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right)^{\mu-1}\right\}>\eta\ .$$

if$\Re e\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}-\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right\}\geq-\delta\frac{(1-\eta)}{2\beta}$ (17)

For some $\eta(0\leq\ \eta<1$ then $\Re e\left\{\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right\}>\eta$ (18)

and$\Re e\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right\}>\frac{\eta(2\beta+\delta)-\delta}{2\beta}$ (19)

Proof: let $(z)=\frac{1}{(1-\eta)}\left\{\left(\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}-\eta\right)\right\}$, then $q(z)$ is analytic function in$\mathbb{D}$ with $q(0)=1$. Now , if we setting $\varphi(z)=\frac{\chi_p^{n+1}(\alpha,\beta)g(z)}{\chi_p^{n}(\alpha,\beta)g(z)}$, then form (7), we have $\Re e\{\varphi(z)\}>\delta$ and $\delta\in\mathbb{D}$.A easy calculation show that

$$(1-\eta)q'(z)=\frac{\chi_p^{n+1}(\alpha,\beta)g(z)\left(\chi_p^{n+1}(\alpha,\beta)f(z)\right)'-\chi_p^{n+1}(\alpha,\beta)f(z)\left(\chi_p^{n+1}(\alpha,\beta)g(z)\right)'}{\left(\chi_p^{n+1}(\alpha,\beta)g(z)\right)^2}$$

Solving above and using the relation (4), we have$\alpha z(1-\eta)q'(z)=\frac{\chi_p^{n+1}(\alpha,\beta)g(z)\chi_p^{n}(\alpha,\beta)f(z)-\chi_p^{n+1}(\alpha,\beta)f(z)\chi_p^{n}(\alpha,\beta)g(z)}{\left(\chi_p^{n+1}(\alpha,\beta)g(z)\right)^2}$

Now by the definition of $\varphi(z)$, we get $\frac{\alpha z(1-\eta)q'(z)}{\beta}\varphi(z)=\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}-\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}=\varphi\left(q(z),zq'(z)\right)$, where $\Psi(r,s)\ =\frac{(1-\eta)s\varphi(z)}{\beta}$.

So by the our assumption

$$\{\Psi(q(z),zq'(z);z\in\mathbb{D})\}\subset\Omega=\left\{\omega\in C:\Re e(\omega)>-\frac{(1-\eta)\delta)}{2\beta}\right\}$$

$$\Re e\{\Psi(\mathrm{i}r_2,s_1)\}=\frac{r_1(1-\eta)\Re e\{\varphi(z)\}}{\beta}\leq\ -\frac{(1-\eta)\delta}{2\beta}$$

This is shows that $\Psi(\mathrm{i}r_2,s_1)\notin\Omega$ for each $z\in\mathbb{D}$. Hence by the lemma [1] we get $\Re e\{q(z)\}>0\ (z\in\mathbb{D})$. This is the proof of (18). Now for the proof of (19), $\Re e\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)g(z)}\right\}=\Re e\left\{\frac{\chi_p^{n}(\alpha,\beta)f(z)}{\chi_p^{n}(\alpha,\beta)f(z)}-\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)f(z)}\right\}+$ $\Re e\left\{\frac{\chi_p^{n+1}(\alpha,\beta)f(z)}{\chi_p^{n+1}(\alpha,\beta)g(z)}\right\}\geq\ -\frac{(1-\eta)\delta}{2\beta}+\eta$

$\geq\frac{\eta(2\beta+\delta)-\delta}{2\beta}$which is the complete proof of (19).

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