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New technique of two numerical methods for solving integral equation of the second kind

M. A. Abdou[1], A. A. Abbas[2]
Mathematics Department, Faculty of Education, Alexandria University
Mathematics Department, Faculty of Science, Benha University
Corresponding Author: M. A. Abdou

Abstract: Here, we use new technique ofTrapezoidal Rule and Simpson's Rule**,** as two numerical approximation methods to solve the integral equations of type Fredholm and Volterrs with conuiouskernel.Some numerical examples are considered and the error of the method is computed. Comparizonbetween the solution of ordinary differential and the corresponding solution of integral equation, numerically is obtained.

Key Wards: Volterra integral equation, Trapezoidal Rule, Simpson's Rule, approximate method, the error of the method.

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I. INTRODUCTION

Many problems of mathematical physics, engineering and contact problemsin the theory of elasticity lead to integral equations (Abdou, 2002; Abdou,131, 2002; Abdou, 2003).Their solutions can be obtained analytically, using the theory developed by (Mushkelishvili,1953). The books edited by (Green,1969); Hochstadt,1971; Golberg.ed,1979 and Tricomi,1985) contain many different methods to solve the integral equations analytically. The book edited by (Golberg.ed,1990) contain extensive literature surveys on both approximate analytical and purely numerical techniques. The interested reader should consult the fine expositions by (Atkinson , 1976, 1997); (Delves and Mohamed,1985) and (Linz,1985) for numerical methods. (Abdou, 137,2003) obtained a Fredholm integral equation of the first kind with singular kernel, when the mixed problem of continuous media with boundary conditions specified on a circle is studied. (Abdou131,2002) obtained and solved the Fredholm integral equation of the second kind, when the kernel takes the Weber–Sonin integral form. The integral equation is investigated, in this case, from the semi-symmetric Hertz problem of two different materials in three-dimensional. Also the resolving of Fredholm integral equation of the second kind, and others cases are studied by (Abdouand Nasr, 2003).

Here, in this paper, we discuss the solution of integral equation using Trapezoidal Rule and Simpson's Rule**.** Moreover, we compare the numerical results with the corresponding results of ordinary differential equation

II. SOLUTION OF VOLTERRA EQUATION USING TRAPEZOIDAL RULE

To solve Volterra integral equation

$$\varphi(x) = g(x) + \int_a^x k(x,y)\varphi(y)dy \qquad (1)$$

numerically by another way that call Trapezoidal rule, we apply the basic formula of the trapezoidal rule , see (Atkinson, 1976, 1997) we obtain

$$\int_a^{x_j} k(x_j, y)\emptyset(y)dy = h_j\,(\frac{1}{2}k_{j,0}y_0 + \sum_{i=1}^{j-1} k_{j,i}x_i + \frac{1}{2}k_{j,j}y_j) + g_j \qquad , j = 1,2,3,\dots,N\,. \quad (2)$$

Here, in (2), we divide the interval $[0, x]$ to N parts, where we assume
$0 = x_0 \le t_1 \le x_2 \le \cdots \le x_i \le \cdots \le x_n = x$;
Also, we wtite
$\Delta y = (x_n - a)/n;\ y_0 = a,\ y_j = a + j\,\Delta y\ = \mathrm{y}_0 + j\,\Delta y;\ j = 1,2,\dots,n-1.$
Moreover, to find $\varphi(x)$, we consider
$x_0 = y_0 = a \text{ and } x = x_n = y_n;\ x_i = x_0 + i\,\Delta y = a + i\,\Delta y = y_i$
$i,e\ \ x_i = y_i,\ 1 \le i \le n-1.$

Using the following definitions $g(x_i) = g_i, k(x_i, y_j) \equiv k_{ij}$, *we have*

$$\int_a^x k(x,y)\varphi(y)dy \approx \Delta y[\frac{1}{2}k(x,y_0)\varphi(y_0) + k(x,y_1)\varphi(y_1) + \ldots$$

$$+ \mathrm{k}(\mathrm{x},\mathrm{y}_{\mathrm{n-1}})\varphi(y_{n-1}) + \frac{1}{2}k(x,y_n)\varphi(y_n)] \quad , \quad (3)$$

$$\Delta \mathrm{y} = \frac{\mathrm{y}_\mathrm{j} - a}{j} = \frac{x-a}{n} \ , \ \mathrm{y}_\mathrm{j} \le x \quad , \quad \mathrm{j} \ge 1 \ , \quad \mathrm{x} = \mathrm{x}_\mathrm{n} = y_n$$

III. SOLUTION OF VOLTERRA EQUATION USING SIMPSON'S RULE :

Consider Simpson's Rule

$$A = \int_a^x f(y)dy = \frac{h}{3}[f(a) + 2\sum_{i=1}^{(n/2)-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(x)] \qquad (4)$$

Using the result of (3) in (1) we have the approximate solution of (1), for $a \le y \le x$. While, the solution $\varphi(x)=0, \ \forall y>x$. The error of the method can be obtained from the term (n+1).

As the previous section, we divide interval $[a, x]$ into N segment, rewrite the integral term of (1) with $x = x_i$, where $0 = x_0 < x_1 < \cdots < x_i < \cdots < x_N = x$, to get

$$\int_a^{x_i} k(x_i,y)\varphi(y)dy \approx h\sum_{j=0}^{i} w_{ij} k(x_i,y_j)\varphi(y_j) \ = \ \mathrm{h}\sum_{\mathrm{j}=0}^{\mathrm{i}} \mathrm{w}_{\mathrm{ij}} k_{ij}\varphi(y_j) + E_{i,y}(k(x_i,y),\varphi(y)) \qquad (5)$$

$$\{x = x_i = a + ih, \quad \mathrm{h} = (\mathrm{x-a})/\mathrm{n}\ , \quad \mathrm{x}_\mathrm{i} = y_i \ , \quad \mathrm{i} = 2,3,\ldots,\mathrm{n}\}$$

Where, w_{ij} represent the weight function, while $E_{i,y}(k(x_i,y),\varphi(y))$ is the error function.

With the aid of Simpson's third rule and Days rule, we get

$$\varphi_r = g_r + \frac{h}{3}\sum_{j=0}^{r} w_{rj} k[rh, jh]\varphi_j, \quad \mathrm{r} = 2, 4, 6, \ldots \qquad (6)$$

IV. ANALYSIS OF THE ERROR

Consider the integral equation

$$\varphi(t) = f(t) + \int_a^t k(t,s)\varphi(s)ds \quad , \quad 0 \le \mathrm{t} \le \mathrm{T} < 1 \qquad (7)$$

Using a numerical method, we have

$$\int_a^t k(t,s)\varphi(s)ds = \int_a^{t_n} k(t_n,s)\varphi(s)ds \approx h\sum_{i=0}^{t_n} w_{ni} k(t_n,t_i)\varphi(t_i) \qquad (8)$$

The transformation from integral formula to an algebraic formula, we have a local error

$$E(h,t_n) = \int_a^b k(t_n,s)\varphi(s)ds - h\sum_{i=0}^{n} w_{in} k_{in}\tilde{\varphi}_i \qquad (9)$$

The local error depends on its calculations on two important bases:
A) Measuring accuracy in the vicinity of the given function.
B - the base of numerical integration used.
1 Estimation of error and numerical stability:

If the numerical solution grows and slowly increases when compared to error i.e, there is constant error or a slight increase is not comparable to the increase in numerical solution to the exact solution), the method used is poor. On the contrary, the error may increase more rapidly than the numerical solution approaches the exact solution. Therefore, this phenomenon is called numerical instability and this phenomenon is very well known when solving normal differential equations. We will show that this phenomenon also applies when solving integral equations.

Consider the exact solution φ and $\tilde{\varphi}$ is the numeric solution. Hence, the error formula is

$$e(t) = \varphi(t) - \tilde{\varphi}(t) \qquad (10)$$

Subtracting the numerical solution from the exact solution, we get

$$\varphi(t)-\tilde{\varphi}(t)=\int_a^b k(t,s)[\varphi(s)-\tilde{\varphi}(s)]ds-E(t) \qquad (11)$$

Hence, the integral equation of the computational error

$$e(t)=\int_a^b k(t,s)e(s)ds-E(t) \qquad (12)$$

Here, the known functions represent the kernel and the function of transformation.
Let the integral operator of the error is

$$Ke=\int_a^b k(t,s)e(s)ds$$

Then, we have

$$e(t)=Ke-E(t)\Rightarrow(1-K)e=-E$$

$$\therefore \|e\|\le\|(1-K)^{-1}\|\ \|\mathrm{E}\|, \quad \det(1\text{-K})\neq 0$$

$$\|(1-K)^{-1}\|\le\frac{1}{(1-\|K\|)}\Rightarrow\|K\|<1.$$

V. SOME NUMERICAL EXAMPLES

Example (4.1): Find the exact and numerical solution of Volterra equation

$$\varphi(x)=x-\frac{1}{6}x^3+\int_0^x(x-y)\varphi(y)dy \qquad (13)$$

Solution:(1) Analytic solution, using successive approximation method, we have

$\varphi_0(x)=0;\varphi_1(x)=\text{x-}\frac{1}{6}x^3,\ \varphi_2(x)=\text{x-}\frac{1}{120}x^5\ ;\varphi_3(x)x-\frac{1}{5040}x^7;\ \ \varphi_4(x)x-\frac{1}{362880}x^9$,.....

The above results can be adapted in the form

$$\varphi_0(x)=0;\varphi_1(x)=\text{x-}\frac{1}{2.3}x^{2+1},\ \varphi_2(x)=\text{x-}\frac{1}{2.3.4.5}x^{2.2+1}\ ;\varphi_3(x)x-\frac{1}{2.3.4.5.6}x^{2.3+1}$$

$$\varphi_4(x)x-\frac{1}{2.3.4.5.6.7.8}x^{2.4+1};...;\ \varphi_n(x)=x-\frac{1}{(2n+1)!}x^{2n+1} \qquad (14)$$

The exact solution is

$$\varphi(x)=x-\lim_{n\to\infty}\frac{1}{(2n+1)!}x^{2n+1}=x \qquad (15)$$

2) Numeric solution: Applying Trapezoidal Rule and Simpson's rule for n=50, we have

x	Exact solution	Trapezoidal rule	Error	Simpson's rule	Error
0	0	0	0	0	0
0.10	0.10	0.09999332266	0.667734×10^{-5}	0.09999332622	0.667378×10^{-5}
0.20	0.20	0.1999865785	0.0000134215	0.1999999395	0.605×10^{-7}
0.30	0.30	0.2962763764	0.0037236236	0.2962815029	0.0037184971
0.40	0.40	0.3938053147	0.0061946853	0.3937903664	0.0062096336
0.50	0.50	0.4805440703	0.0194559297	0.4805487806	0.0194512194
0.60	0.60	0.5714788227	0.0285211773	0.5714508716	0.0285491284
0.70	0.70	0.6447547764	0.0552452236	0.6447588529	0.0552411471
0.80	0.80	0.7248409743	0.0751590257	0.7248011596	0.0751988404
0.90	0.90	0.7808828822	0.1191171178	0.7808861083	0.1191138917

Table (1)

The numeric results for the exact solution, and Trapezoidal and Simpson's rules with the error in each case

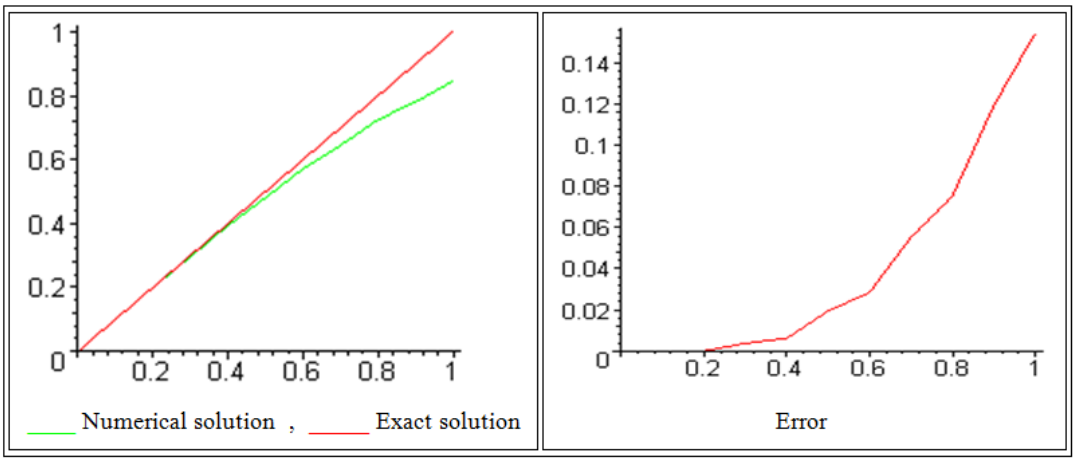

Figure (1)

The relation between the exact and numeric solution (Trap. rule). In addition, the error is computed

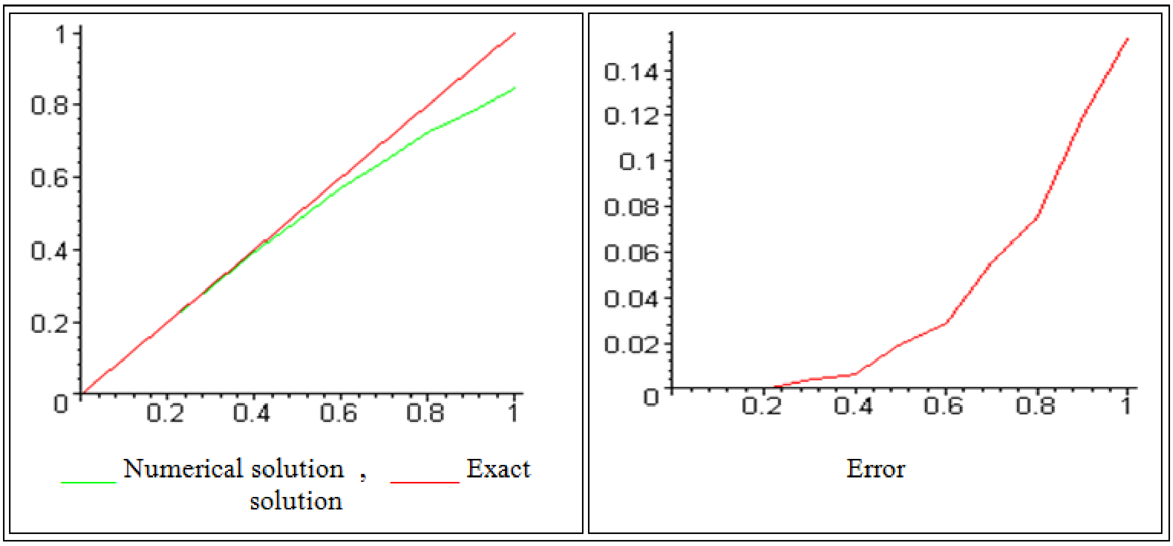

Figure (2)

The relation between the exact and numeric solution (Sim. rule). In addition, the error is computed

Example (2): Consider the Volterra integral equation

$$\phi(t) + \int_0^t e^{x+t}\,\phi(s)ds = e^{2t}(t-2)^2 - 4e^t + t^2 - 2t, \tag{16}$$

It is difficult to obtain, directly the exact solution, using the classical famous

methods. Therefore,write equation (16), in the form

$$u(t) = \frac{f(t)}{P(t)} - \frac{1}{v(t)}\int_0^t \acute{v}(s)\left[\frac{f(s)}{P(s)}\right]ds \tag{17}$$

Here,

$$u(t) = \frac{e^{2t}(t-2)^2 - 4e^t + t^2 - 2t}{e^t} - \frac{1}{v(t)}\int_0^t \acute{v}(s)\left[\frac{e^{2s}(s-2)^2 - 4e^s + s^2 - 2s}{e^s}\right]ds\,. \tag{18}$$

The integrating factor

$$v(t) = exp\left\{\int P(s)Q(s)ds\right\} = \exp\left\{\int e^{2s}ds\right\} = \exp\left(\frac{e^{2s}}{2}\right)$$

The formula (16), yields

$$\phi(t) = e^{2t}(t-2)^2 - 4e^t + t^2 - 2t - \exp\left(\frac{-e^{2t}}{2}\right)\int_0^t \exp\left(\frac{e^{2s}}{2}\right) e^s[e^{2s}(s-2)^2 - 4e^s + s^2 - 2s]ds \,. \quad (19)$$

It is not easy to calculate the latter integration, so, we solve the Volterra integral equation (16) numerically, then an integral part of equation (19) is solved as a part of solving differential equation to find the value of $\phi(t)$. In both we use the famous methods, Trapezoidal method and modified Simpson rule in various times.
At t=0.13

The solution of integral equation

x	Exact	App. Trap.	Error Trap.	App. Simp.	Error Simp.
0	0	0	0	0	0
0.013	-0.025831	-0.0258881	5.71489E-05	-0.02587382	4.28228E-05
0.026	-0.051324	-0.0514398	1.15759E-04	-0.05141051	8.65118E-05
0.039	-0.076479	-0.0766548	1.75817E-04	-0.07661005	1.31051E-04
0.052	-0.101296	-0.1015333	2.37306E-04	-0.10147239	1.76395E-04
0.065	-0.125775	-0.1260752	3.00207E-04	-0.12599751	2.22505E-04
0.078	-0.149916	-0.1502805	3.64497E-04	-0.15018535	2.69348E-04
0.091	-0.173719	-0.1741491	4.30150E-04	-0.17403587	3.16870E-04
0.104	-0.197184	-0.1976811	4.97136E-04	-0.19754903	3.65030E-04
0.117	-0.220311	-0.2208764	5.65421E-04	-0.22072478	4.13777E-04
0.13	-0.2431	-0.243735	6.34970E-04	-0.24356304	4.63041E-04

Table (2)

The numerical solution of the integral equation (16) using Trapezoidal rule and Simpson's rule

The solution of differential equation

x	Exact	App. Trap.	Error Trap.	App. Simp.	Error Simp.
0	0	0	0	0	0
0.013	-0.025831	-0.025887306	5.63065E-05	-0.025916218	8.52183E-05
0.026	-0.051324	-0.051786346	4.62346E-04	-0.051845713	5.21713E-04
0.039	-0.076479	-0.077715248	1.23625E-03	-0.077806698	1.32770E-03
0.052	-0.101296	-0.103692696	2.39670E-03	-0.103817944	2.52194E-03
0.065	-0.125775	-0.129737972	3.96297E-03	-0.129898827	4.12383E-03
0.078	-0.149916	-0.155870969	5.95497E-03	-0.156069342	6.15334E-03
0.091	-0.173719	-0.182112179	8.39318E-03	-0.182350086	8.63109E-03
0.104	-0.197184	-0.208482742	1.12987E-02	-0.208762311	1.15783E-02
0.117	-0.220311	-0.235004424	1.46934E-02	-0.235327904	1.50169E-02
0.13	-0.2431	-0.262439175	1.93392E-02	-0.262624059	1.95241E-02

Table (3)

The numerical solution of the differential equation (19) using Trapezoidal rule and Simpson's rule

Example (3): Find the exact and numerical solution of Volterra equation

$$\varphi(x) = x^2 - \frac{x^5}{4} + \int_0^x xy\,\varphi(y)\,dy$$

Solution:(1) Analytic solution, successive approximation method

$$\varphi_0(x)=g(x_0)=0,\ \ \varphi_1(x)=g(x)+\int_0^x xy\,\varphi_0(y)dy\ =x^2-\frac{x^5}{4}$$

$$\varphi_2(x)=g(x)+\int_0^x xy\,\varphi_1(y)dy\ =\mathrm{x}^2-\frac{1}{28}x^8,\ \ \varphi_3(x)=g(x)+\int_0^x xy\,\varphi_2(y)dy\ =\mathrm{x}^2-\frac{1}{280}x^{11}$$

$$\varphi_4(x)=g(x)+\int_0^x xy\,\varphi_3(y)dy\ =\mathrm{x}^2-\frac{1}{3640}x^{14}$$

The above formula can be adapted in the form

$$\varphi_0(x)0,\ \ \varphi_1(x)=x^2-\frac{x^5}{4},\ \ \varphi_2(x)=\mathrm{x}^2-\frac{1}{4.7}x^{5+3};\qquad \varphi_3(x)=\mathrm{x}^2-\frac{1}{4.7.10}x^{5+3.2}$$

$$\varphi_4(x)=\mathrm{x}^2-\frac{1}{4.7.10.13}x^{5+3.3}\ \ \ldots\ \ \varphi_n(x)=\mathrm{x}^2\ \text{-}\ \frac{1}{4.7.10.13...(3n+1)}x^{3n+2}$$

The exact solution

$$\varphi(x)=\lim_{n\to\infty}\varphi_n(x)=\lim_{n\to\infty}(\mathrm{x}^2\ \text{-}\ \frac{1}{4.7.10.13...(3n+1)}x^{3n+2})\Rightarrow\varphi(x)=\mathrm{x}^2$$

(2) Numeric solution: Applying Trapezoidal Rule and Simpson's rule for n=50, we have

x	Exact solution	Trapezoidal rule	Error	Simpson's rule	Error
0	0	0	0	0	0
0.10	0.0100	0.0100010002	0.10002×10^{-6}	0.01000019876	0.19876×10^{-6}
0.20	0.0400	0.0400080130	0.80130×10^{-6}	0.03999999927	0.73×10^{-9}
0.30	0.0900	0.08941185251	0.00058814749	0.08941543257	0.00058456743
0.40	0.1600	0.1577803842	0.0022196158	0.1577483942	0.0022516058
0.50	0.2500	0.2422437892	0.0077562108	0.2422537459	0.0077462541
0.60	0.3600	0.3413831339	0.0186168661	0.3413136662	0.0186863338
0.70	0.4900	0.4480902067	0.0419097933	0.4481088633	0.04181522872
0.80	0.6400	0.5595191145	0.0804808855	0.554050712	0.0805949288
0.90	0.8100	0.6625406282	0.1474593718	0.6625684723	0.1474315277

Table (4)

The numeric results for the exact solution, Trap rule andSimp. rules with error in each case

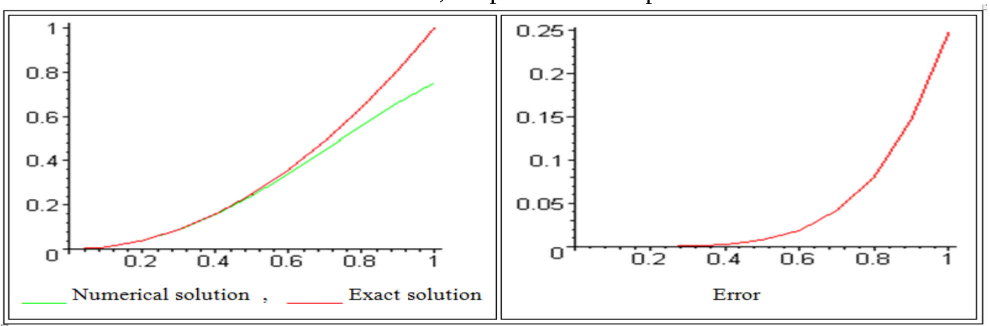

Figure (3)

The relation between the exact and numeric solution using Trap. Rule with the error

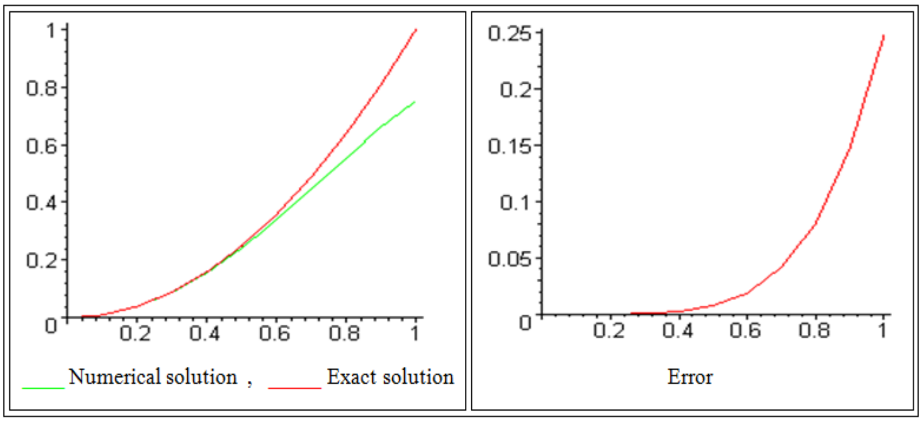

Figure (4)

The relation between the exact and numeric solution using Simp.Rule, with the error.

VI. CONCOLUSION

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