Some Expansion Formulae For Thealeph(\(\aleph\)). Function

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Abstract: In the present paper, the author has established two expansion formula of Aleph \(\aleph\)-Function.

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I. INTRODUCTION

The \(\aleph\)- function introduced by Suland et.al. [3] defined and represented in the following form:

\[
\aleph[z] = \aleph_{m,n,r}^{p,q,i} \left[ z \left| \begin{array}{c} (a_j, \alpha_j), \tau_j, (a_j, \beta_j)_{n+1, p_i} \\ (b_j, \beta_j)_{m+1, q_i} \\ \end{array} \right. \right] \\
= \frac{1}{2\pi i} \int_{L} \theta(s) z^{s} ds
\]  

\(1.1\)

Where \(\omega = \sqrt{-1}\):

\[
\theta(s) = \prod_{j=1}^{m} \Gamma(b_j - \beta_j, s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j, s) \sum_{j=1}^{q} \tau_j \left\{ \left[ \begin{array}{c} (a_j, \alpha_j), \tau_j, (a_j, \beta_j)_{n+1, p_i} \\ (b_j, \beta_j)_{m+1, q_i} \\ \end{array} \right. \right. \\
\]  

\(1.2\)

We shall use the following notations:

\(A^* = (a_j, \alpha_j)_{m,n,r}, \tau_j, (a_j, \beta_j)_{n+1, p_i}; \ B^* = (b_j, \beta_j)_{m+1, q_i} \)

II. EXPANSION FORMULA

First Formula

\[
\aleph_{m,n,r}^{p,q,i} \left[ \eta \omega \right] = \aleph_{m,n,r}^{p,q,i} \left[ \frac{1 - \eta^{1/\beta}}{\beta} \right] \sum_{r=0}^{\infty} \frac{1}{r!} \aleph_{m,n,r}^{p,q,i} \left[ \omega \left| A^* \right| \left( \eta^{1/\beta} \right)^{r} \right]
\]  

\(2.1\)

Where \(\eta\) is written for \(m = 1\) and for \(m > 1, |\eta^{1/\beta} - 1| < 1; \arg(\eta \omega) = \beta; \arg(\eta^{1/\beta}) + \arg \omega\) and \(|\arg(\eta^{1/\beta})| < \frac{\pi}{2}\).

Proof: R.H.S. = \(\aleph_{m,n,r}^{p,q,i} \left[ \eta \omega \right] \)

\[
\aleph_{m,n,r}^{p,q,i} \left[ \frac{1 - \eta^{1/\beta}}{\beta} \right] \sum_{r=0}^{\infty} \frac{1}{r!} \aleph_{m,n,r}^{p,q,i} \left[ \omega \left| A^* \right| \left( \eta^{1/\beta} \right)^{r} \right]
\]  

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\[ h^{n_\beta} \sum_{r=0}^{\infty} \left[ \frac{1-n^{\beta/r}}{r!} \right]^{\gamma} \int_{L} \frac{1}{2\pi i} \left\{ \prod_{j=m+1}^{n} \Gamma(1-b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j + \alpha_j s) \right\} ds \]

Changing the order of integration and summation under the integral sign

\[ \frac{1}{2\pi i} \int_{L} \theta(s) \omega^{s} \left[ h^{n_\beta} \sum_{r=0}^{\infty} \frac{1-n^{\beta/r}}{r!} \Gamma(r+b_1 - \beta_1 s) \right] ds \]

\[ \frac{1}{2\pi i} \int_{L} \theta(s) \omega^{s} \left[ h^{n_\beta} \sum_{r=0}^{\infty} \frac{1-n^{\beta/r}}{r!} (b_1 - \beta_1 s), \Gamma(b_1 - \beta_1 s) \right] ds \]

\[ \frac{1}{2\pi i} \int_{L} \theta(s) \omega^{s} \left[ h^{n_\beta} \left[ (1-\eta) \eta^{(-\beta_1 s)} \right] \Gamma(b_1 - \beta_1 s) \right] ds \]

\[ \sum_{r=1}^{\infty} \left[ \frac{\eta^{n_\beta}}{r!} - 1 \right]^{\gamma} \int_{L} \left( \prod_{j=m+1}^{n} \Gamma(1-b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j + \alpha_j s) \right) (\omega \eta)^{s} ds = \text{L.H.S.} \]

Second Formula

\[ N^{m,n}_{p_1,q_1,\tau,\zeta} \left[ \eta \omega \right] = h^{n_\beta} \sum_{r=0}^{\infty} \frac{\eta^{n_\beta} - 1}{r!} N^{m,n}_{p_1,q_1,\tau,\zeta} \left[ \omega \right]^{n_\beta} \left( b_j, \beta_j, \gamma, \alpha_j, \gamma, \tau, \zeta \right) \]  \hspace{1cm} (2.2)

Where \( q > m, |\eta^{n_\beta} - 1| < 1; \arg(\eta \omega) = \beta_q \arg(\eta^{n_\beta}) + \alpha_q \omega \) and \( |\arg(\eta^{n_\beta})| < \frac{\pi}{2} \).

Proof: R.H.S. = \[ h^{n_\beta} \sum_{r=0}^{\infty} \frac{\eta^{n_\beta} - 1}{r!} \int_{L} \left( \prod_{j=m+1}^{n} \Gamma(1-b_j + \beta_j s) \prod_{j=1}^{n} \Gamma(1-a_j + \alpha_j s) \right) ds \]

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Changing the order of integration and summation under the integral sign

\[
\frac{1}{2\pi i} \int_{L} \theta(s) \omega^i \left[ \sum_{r=0}^{\infty} \frac{\eta^r/\beta_k}{r! (1-r-b_q + \beta_q s)} \right] ds
\]

\[
= \frac{1}{2\pi i} \int_{L} \theta(s) \omega^i \left[ \sum_{r=0}^{\infty} \frac{\eta^r/\beta_k}{r! (1-b_q + \beta_q s, \Gamma(1-b_q + \beta_q s)} \right] ds
\]

\[
= \frac{1}{2\pi i} \int_{L} \theta(s) \omega^i \left[ \sum_{r=0}^{\infty} \frac{\eta^r/\beta_k}{r! (1-b_q + \beta_q s)} \right] \left[ \frac{1}{\eta^r/\beta_k \Gamma(1-b_q + \beta_q s, \Gamma(1-b_q + \beta_q s)} \right] ds
\]

\[
= \frac{1}{2\pi i} \int_{L} \frac{\theta(s) \omega^i}{\Gamma(1-b_q + \beta_q s, \Gamma(1-b_q + \beta_q s)} \left[ \frac{1-(1-\eta)^r/\beta_k}{r! (1-b_q + \beta_q s, \Gamma(1-b_q + \beta_q s)} \right] \left[ \frac{b_q}{\beta_k} \right] ds
\]

\[
= \frac{1}{2\pi i} \int_{L} \frac{\theta(s) \omega^i}{\Gamma(1-b_q + \beta_q s, \Gamma(1-b_q + \beta_q s)} \left[ \frac{1}{\eta^r/\beta_k \Gamma(1-b_q + \beta_q s, \Gamma(1-b_q + \beta_q s)} \right] ds
\]

For \( \tau_i = 1, r = 1 \) in (2.1),(2.2), we get the results in terms of Fox’s H-function [1].

REFERENCES


