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Some Curvature Properties of Sasakian Manifolds

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Abstract: - *In this paper, we established several curvature properties of a Sasakian manifold of dimension* $n \geq 5$. *The Bochoner curture tensor B of a Sasakian manifold* M^{2n+1} *with structure* (φ, ξ, η, g) *is introduced as an analogue of the Weyl conformal curvature tensor of a Reimannian manifold (see Matsumoto and Chuman [3]) We obtained many identities containing components of Bochoner curture tensor. We studied some geometrical implication of vanishing of the contact Bochoner curture tensor in Proposition 2.1 and 2.2*
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I. Introduction:

A differential manifold M^{2n+1} is said to have a (φ, ξ, η, g)-structure if it admits an endomorphism φ of the tangent spaces, a vector field ξ and a 1-form η satisfying

(1.1) $$\eta(\xi) = 1$$

and

(1.2) $$\varphi^2 = -I + \eta \otimes \xi$$

where I denotes the identity transformation. It is easily seen that φ satisfies

(1.3) $$\varphi\xi = 0 \text{ and } \eta \circ \varphi = 0$$

That is, φ has rank 2n. The notion of an almost contact structure and a (φ, ξ, η)-structure are equivalent. In this sense we sometimes refer to an almost contact structure (φ, ξ, η).

We also see that M^{2n+1} admit a special Riemannian metric called a compatible metric such that

(1.4) $$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

M^{2n+1} with (φ, ξ, η)-structure and then metric (1.4) is said to have (φ, ξ, η, g)-structure or an almost contact metric structure (φ, ξ, η, g).

The fundamental 2-form Φ of an almost contact metric structure (φ, ξ, η, g) is defined by

(1.5) $$\Phi(X, Y) = g(\varphi X, Y).$$

Φ is skew-symmetric because of (1.2), (1.3) and (1.4). An odd dimensional Euclidean space R^{2n+1}, a hypersurface in an almost complex manifold, especially an odd dimensional sphere, a product manifold $M^{2n} \times R$ of an almost complex manifold and real line are examples of almost contact metric manifolds.

An almost contact manifold M^{2n+1} is said to be normal if an almost complex structure $M^{2n+1} \times R$ is normal, that is,

(1.6) $$[\varphi, \varphi](X, Y) + d\eta(X, Y)\xi = 0$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion tensor for φ.

An almost contact metric structure is said to be a contact structure if

(1.7) $$\Phi(X, Y) = d\eta(X, Y)$$

A Sasakian manifold is an almost contact metric manifold satisfying (1.6) and (1.7). But it is well known that

(1.8) $$(\nabla_X\varphi)Y = -g(X,Y)\xi + \eta(Y)X$$

is the necessary and sufficient condition for an almost contact metric manifold to be a Sasakian manifold. See Blair[1] . In a Sasakian manifold we have

(1.9) $$\nabla_X\xi = \varphi X$$

By (1.8), (1.9) and the Ricci identity for ξ we have

(1.10) $$R_{kji}{}^{h}\xi^{i} = \delta_k^h\eta_j - \delta_j^h\eta_k$$

or

(1.11) $$R_{kji}{}^{h}\eta_h = \eta_k g_{ji} - \eta_j g_{ki}$$

By applying δ_h^k to (1.10), we have

(1.12) $$R_{ji}\xi^{i} = 2n\eta_j$$

Let M^{2n+1} be a Sasakian manifold. The sectional curvature of the section spanned by X and φX which are orthogonal to ξ is called a φ-sectional curvature .A Sasakian manifold of constant φ-sectional curvature c is called a Sasakian space form M^{2n+1}(c). The necessary and sufficient condition for a Sasakian manifold M^{2n+1} $(2n+1\geq 5)$ to be a Sasakian space form M^{2n+1}(c) is that the curvature tensor has the following form:

(1.13) $$R(X,Y)Z = \frac{c+3}{4}(g(Y,Z)X - g(X,Z)Y) + \frac{c-1}{4}(\eta(X)\eta(Z)Y$$
$$-\eta(Y)\eta(Z)X + g(X,Z)\eta(y)\xi - g(Y,Z)\eta(X)\xi$$
$$+ g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y - 2g(\varphi X,Y)\varphi Z).$$

If the curvature tensor is of the form (1.13), the Ricci tensor R_{ji} and the scalar curvature S are given by

(1.14) $$R(X,Y) = \frac{n(c+3)+c-1}{2}g(X,Y) - \frac{(n+1)(c-1)}{2}\eta(X)\eta(Y)$$

and

(1.15) $$S = \frac{1}{2}[n(2n+1)(c+3) + n(c-1)].$$

An odd dimensional sphere S^{2n+1} and odd dimensional Euclidian space R^{2n+1} and the product bundle (R, CD^n), where CD^n is a simply connected homogeneous complex domain with constant holomorphic sectional curvature having value less than or equal to zero and R is the real line, are examples of Sasakian space forms.

By generalizing (1.14), we call a Sasakian manifold M^{2n+1} C- Einstein if Ricci tensor R_{ji} of M^{2n+1} is of the form (see Kon[2])

(1.16) $$R_{ji} = ag_{ji} + b\eta_j\eta_i,$$

where $$a + b = 2n.$$

Remark : The second Bianchi identity reduces to

(1.17) $$\nabla_j S - 2\nabla_i R_j^i = 0$$

From (1.16), the scalar curvature is expressed by

(1.18) $$S = (2n+1)a + b = 2n(a+1).$$

By putting (1.16) and (1.18) into (1.17), we have

(1.19) $$(2n-2)\nabla_j a + 2\eta_j \xi^i \nabla_i a = 0,$$

Since (1.9) and (1.3) hold. Applying ξ^j, we see

(1.20) $$2n\xi^j \nabla_j a = 0.$$

Thus if $n > 1$, a and b in (1.16) are necessarily constants, because (1.19) with the second term replaced by (1.20) gives

$$(2n-2)\nabla_j a = 0.$$

II. Contact Bochner curvature tensor

From this section Sasakian manifolds always have dimension $n \geq 5$. The contact Bochoner curvature tensor B of a Sasakian manifold M^{2n+1} with structure
tensor (φ, ξ, η, g) is introduced as an analogue of the Wey1 conformal curvature tensor of a Riemannian manifold. (See Matsumoto and Chuman) [3] But we do not know as to what kind of non-trivial transformation leaves the contact Bochoner curvature tensor invariant.

(2.1) $$\begin{aligned} B_{kji}{}^h = R_{kji}{}^h &+ (\delta_k{}^h - \eta_k \xi^h) L_{ji} - (\delta_j{}^h - \eta_j \xi^h) L_{ki} \\ &+ L_k{}^h (g_{ji} - \eta_j \eta_i) - L_j{}^h (g_{ki} - \eta_k \eta_i) \\ &+ \varphi_k{}^h M_{ji} - \varphi_j{}^h M_{ki} + M_k{}^h \varphi_{ji} - M_j{}^h \varphi_{ki} \\ &- 2(\varphi_{kj} M_i{}^h + M_{kj} \varphi_i{}^h) \\ &+ (\varphi_k{}^h \varphi_{ji} - \varphi_j{}^h \varphi_{kj} - 2\varphi_{kj} \varphi_i{}^h), \end{aligned}$$

where

(2.2) $$L_{ji} = \frac{1}{2(n+1)}[-R_{ji} - (L+3) g_{ji} + (L-1)\eta_j \eta_j],$$

(2.3) $$L_j{}^t = L_{jt} g^{ti},$$

(2.4) $$L = g^{ji} L_{ji},$$

(2.5) $$M_{ji} = -L_{jt} \varphi_i{}^t,$$

(2.6) $$M_j{}^i = M_{jt} g^{ti}$$

From (2.2) and (2.4) it follows that

(2.7) $$L = -\frac{S + 2(3n+2)}{4(n+1)},$$

where S is the scalar curvature of M^{2n+1}
Applying (1.12) to (2.2), we have

(2.8) $$L_{ji} \xi^i = -\eta_j$$

which together with (2.5) yields

(2.9) $$M_{jt} \varphi_i{}^t = L_{ji} + \eta_j \eta_j$$

The following identities are easily verified

(2.10) $$B_{kji}{}^h + B_{jki}{}^h = 0$$

(2.11) $$B_{kji}{}^h + B_{jik}{}^h + B_{ikj}{}^h = 0,$$

(2.12) $$B_{tji}{}^t = 0,$$

(2.13) $$B_{kjih} + B_{kjhi} = 0$$

(2.14) $$B_{kjih} = B_{ihkj},$$

(2.15) $$B_{kji}{}^{h}\eta_h = 0,$$

(2.16) $$B_{kjt}{}^{h}\varphi_i{}^{t} = B_{kji}{}^{t}\varphi_t{}^{h},$$

(2.17) $$B_{kji}{}^{h}\varphi^{kj} = 0$$

Since the vanishing of the Wey1 conformal curvature tensor has an important geometric meaning, next we will study some geometric implications of the vanishing of the contact Bochoner curvature tensor, i.e., B =0.

PREPOSITION 2.1

Let M^{2n+1} be a Sasakian manifold. If M^{2n+1} has constant $\varphi-$sectional curvature, .then M^{2n+1} is C-Einstein and the contact Bochoner curvature tensor B vanishes.

Proof : The first part was already observed in (1.14), so we will just prove the second part. By using (1.14) and (1.15) we have

$$L = -\frac{nc+3n+4}{4}$$

and

$$L_{ji} = -\frac{c+3}{8}g_{ji} + \frac{c-5}{8}\eta_j\eta_i,$$

Thus

$$M_{ji} = -\frac{c+3}{8}\varphi_{ji},$$

which, substituted in (2.1), gives the result.

The converse of Proposition 2.1 is given in the next Proposition 2.2.

PREPOSITION 2.2

Let M^{2n+1} be a Sasakian manifold. If the contact Bochoner curvature tensor B vanishes and M^{2n+1} is a C-Einstein, then M^{2n+1} has a constant $\varphi-$sectional curvature.

Proof : Since M^{2n+1} is C-Einstein, Ricci tensor is expressed by

$$R_{ij} = ag_{ji} + b\eta_j\eta_i$$

where a and b are necessarily constants such that $a+b=2n$. Thus the scalar curvature $S=(2n+1)a+b$ is constant. By using (2.1), we can compute $R_{kji}{}^{h}$ which has the form (1.13) with

$$c = \frac{2na+4a-3n^2-5n+2}{(n+1)(n+2)}$$

References:

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