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Categorization of the Behaviour of A Duffing Oscillator (Subjected To Periodic Excitations) Using Gram-Schmidt Orthogonalized Lyapunov Exponents

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Abstract
A duffing system can respond chaotically to harmonic excitation under the effect of drive parameters, according to the literature, but there is a paucity of materials on how it responds to periodic excitations under the impact of drive parameters. This research is motivated by a desire to learn more about the oscillator's response to periodic excitations. Gram-schmidt orthogonalization was used to simulate and estimate average Lyapunov exponents to characterize the behavior of a duffing oscillator under periodic excitations when constant time step Runge kutta fourth order algorithms were used to solve the periodically excited duffing oscillator using very close initial conditions at constant drive parameter values with amplitude ranging from 0.07 to 1.5.
To validate the PYTHON codes used in this research, the simulated results for the selected periodic excitations were used to create characterization graphs showing the behavior of each periodic excitation (Square wave, Triangle wave, Sawtooth wave) with reference to the harmonic excitation for the selected drive parameter combinations.
For damping coefficient of 0.0168 and 0.168 the proportion of chaos are (85.7%, 82.5%, 74.4% and 82.4%) and (15.5% ,15.6%,13.6%, and 27.4%);(88.4%, 71.5%, 72.5% and 72.8%) and(14.8%, 13.1%, 11.3% and 22.7%); (80.9%, 77.1%, 72.2% and 80.1%) and (15.6%, 15.2%, 13.1% and 28.9%)for harmonic, square, triangle and sawtooth at equilibrium positions (-1,0), (0,0) and (1,0) respectively. The characterization graph generated can be used to predict how a duffing oscillator will behave when subjected to the selected excitations.
Keywords: Periodically excited oscillator, Characterization, Duffing Oscillator, nonlinear system, Lyapunov exponents, Gram Schmidt orthogonalization.

I. Introduction

It can be difficult to study and understand the behavior of nonlinear dynamical systems, which is why the nonlinear duffing oscillator, which is a simplified model of the real nonlinear dynamical system, was created [7]. It provides the researcher or engineer who is interested in researching the behavior of nonlinear systems with an easy-to-work-on frame work for having a grasp of the numerous inherent mechanisms responsibilities[2]. According to published research, when a nonlinear system is activated by an external force or is subjected to an extra perturbation, its dynamical behavior can be altered [9].

Many studies have been conducted in previous years on how nonlinear duffing oscillators behave when subjected to harmonic (sine wave) excitations only, but what interests the authors of this journal is the study of the behavior of the nonlinear system when subjected to periodic excitations, of which three (Square wave, Sawtooth, and Triangle wave) are of particular interest [9]. The highly correlated fourier series were used to calculate these periodic waveforms [8].

The capacity of a nonlinear duffing oscillator to retain its dynamical properties even when subjected to periodic excitations has made it a good fit for secure communication and digital applications [9].Furthermore, the Duffing oscillator is considered one of the prototypes for nonlinear dynamics systems [11]. Wawrzynski [11] described a duffing oscillator to be a body of mass that is being suspended on a parallel combination of a dashpot (cubic stiffness) and a linear springhaving a nonlinear restoring force subjected to harmonic excitations.

Lyapunov exponents have long been regarded to be a reliable tool for describing the behavior of nonlinear dynamical systems [5]. The Lyapunov exponent, which Wikipedia(2013) defines as a quantity that characterizes the rate of separation of infinitesimally close trajectories, is crucial in the characterization of a nonlinear system's behavior because it uses the measurement of the largest Lyapunov exponent in the characterization method [5].Souza-Machado [10] conducted a study to demonstrate how lyapunov exponents can be used as a characterizing tool in the determination of chaos, and these authors picked up on the idea presented in the experiment and conducted their own research on a damped, periodically excited duffing oscillator. When the greatest Lyapunov exponent is positive [4,13,14], it is recognized to imply the presence of chaos in the system [12].The biggest positive Lyapunov exponent may be derived using several methods, however the one used in this work is Gram-Schmidt orthogonalization [13], which is a procedure that aids in the derivation of the largest Lyapunov exponent [5].

The authors used the positive maximum positive Lyapunov exponents to calculate the number of chaotic points [12] for a set of nodal points using drive parameters.

The behaviour of a duffing oscillator under harmonic excitations has been reported by many established literatures over the years but there is a dearth of exploration on the behaviour of a periodically excited duffing oscillator and this research paper is written to address that lacuna.

The following is a breakdown of the four sections of this article paper: 1. A brief overview of the study's background. 2. The research methodology used in this study. 3. Discussion and results 4. Conclusions.

II. METHODOLOGY

Over the years there has been numerous study carried out on harmonically excited duffing oscillator as a model for nonlinear system due to the ability of the oscillator to maintain its nonlinear dynamical system's attributes. The governing equation of the duffing oscillator is given by equation (1) below: [6]

$$\ddot{d} + \gamma\dot{d} - \frac{d}{2}(1 - d^2) = A_o \sin(\omega_d t) \quad (1)$$

where $A_o$is the forcing amplitude, ω_d is the drive frequency, and γ represents the damping coefficient. In order to simulate the governing equation of the duffing oscillator numerically using the computer (as it will be practically impossible to solve the second order nonlinear ordinary differential equation manually) by employing Runge-kutta fourth order scheme requires transformation of the second order nonlinear ODE into a pair of two first order Differential equations (2) and (3) [8].

$d = d_1$ (displacement) (2)

$d_2 = \dot{d}_1$ (velocity, first O.D.E) (3)

After splitting into two first order differential equations then the constant time Rungekutta fourth order scheme method makes use of the pair of first order to solve for the corresponding displacement and velocity for a particular time step and iteration is done for the numerous number of time steps.

2. Fourth- Order Runge Kutta Scheme

$$v_{i+1} = v_i + \frac{h}{6}[M_1 + 2(M_2 + M_3) + M_4] \quad (5)$$

$$M_1 = f(d_i, v_i) \quad (6)$$

$$M_2 = f\left(d_i + \frac{h}{2}, v_i + \frac{hM_1}{2}\right) \quad (7)$$

$$M_3 = f\left(d_i + \frac{h}{2}, v_i + \frac{hM_2}{2}\right) \quad (8)$$

$$M_4 = f(d_i + h, v_i + hM_3) \quad (9)$$

III. STUDY PARAMETERS

This study made use of the following parameters in the course of this research and they are defined by excitation frequency of 1, forcing amplitude ranging from 0.07 to .15, these are investigated for the three selected periodic wave function at nodal points (resolutions) ranging from 51 to 5001 along the amplitude axis for 10 number of counts. These simulations were carried out for the aforementioned parameters at damping coefficients of 0.168 and 0.0168 with fixed simulation time step h= $Tp_{/500}$ for an excitation period of (T=2π). Three sets of equilibrium positions (-1,0),(0,0) and (1,0) were investigated and the simulation was executed for 20 excitation periods including 10-periods of transient and 10-periods of steady solutions.

The Gram-Schmidt orthogonalization was used to determine the Lyapunov exponents[1] by using the Rungekutta fourth order to transform the functions associated with a unit length in orthogonal axes (Xo,Yo) taken as (1,0)[1] using 10 simulation time steps and this is done over steady simulation period after which the simulation is done continuously while the average of all the Lyapunov exponents across the period of the simulation will be calculated and be used to determine the behaviour of the dynamical system under drive parameters that are selected for the simulation process.

A numerical simulation of the forgoing was performed using Rungekutta fourth order algorithms [9] by writing PYTHON subroutines for the simulation using a laptop with specifications: processor INTEL (R) Core (TM) i5-3340M; CPU @ 2.70GHz; Random Access Memory (4Gb) and 64-bit operating system.

Validation cases

The underlisted parameters were used to test run the PYTHON codes for this study for the harmonic excitations and the results was compared with that of established literature written by Dowell [3]. The Poincare section and phase plot trajectory obtained for the validations case at (ω=1.0, $P_0 = 0.21$, γ=0.168) corresponds visually with figures (1) and (3) respectively. Also, the Poincare map of (ω=1.0, $P_0 = 0.19$, γ=0.0168) and corresponds visually with figure (5)

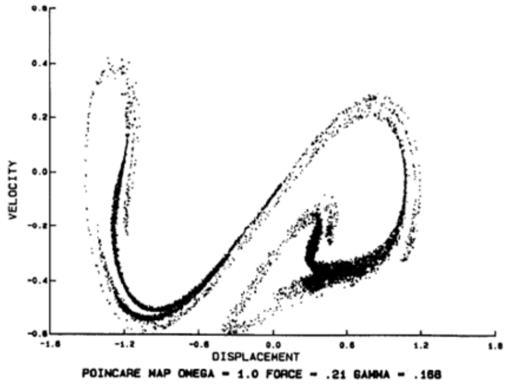

Figure 1: Poincare map [3].

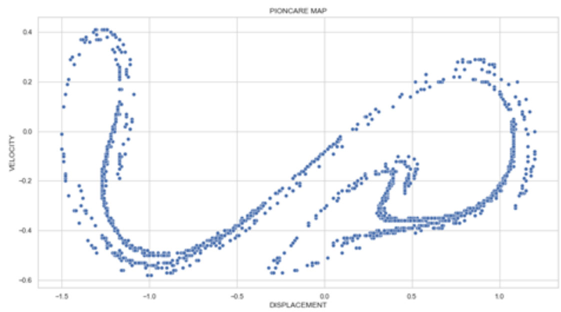

Figure 2: Poincare map from Python simulation.

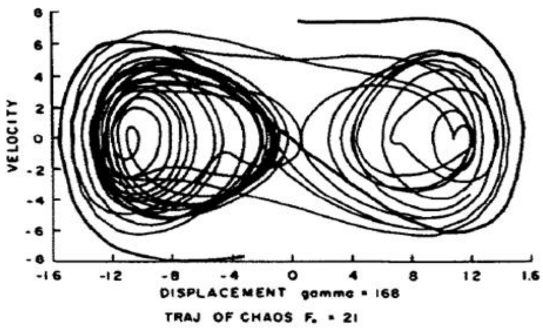

Figure 3: Phase plot trajectories [3].

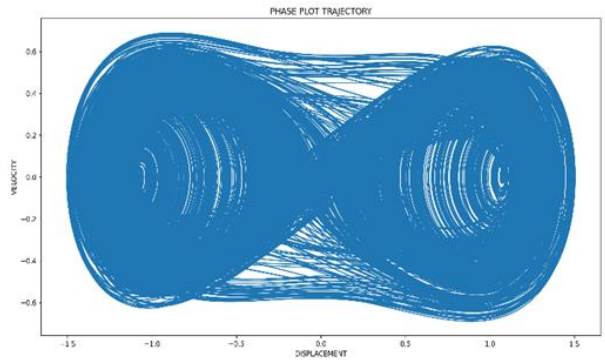

Figure 4: Phase plot trajectories.

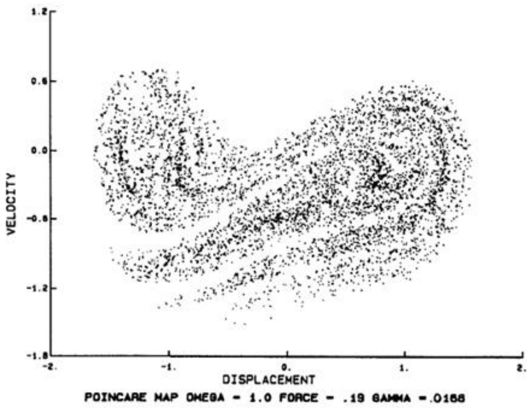

Figure 5: Poincare map [3].

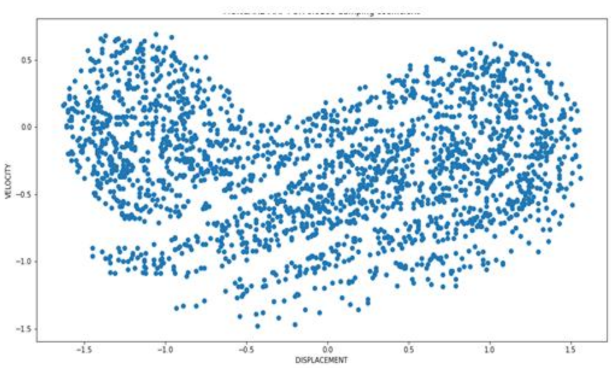

Figure 6: Poincare map from Python simulation.

IV. RESULTS AND DISCUSSIONS

Results

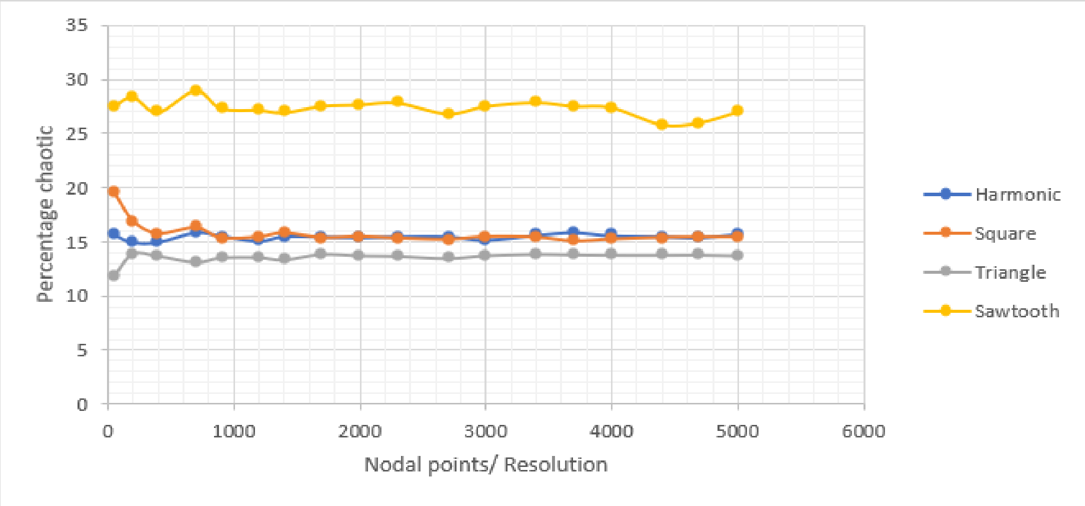

Figure 7: (d_i, v_i) = (-1,0) γ = 0.168.

Given above in figure 7 is the graphical representation of theaverage proportion of chaotic points given quantitively to be(15.5% ,15.6%,13.6%, and 27.4%) for harmonics, square , triangle and sawtooth wave respectively for the above drive parameters.

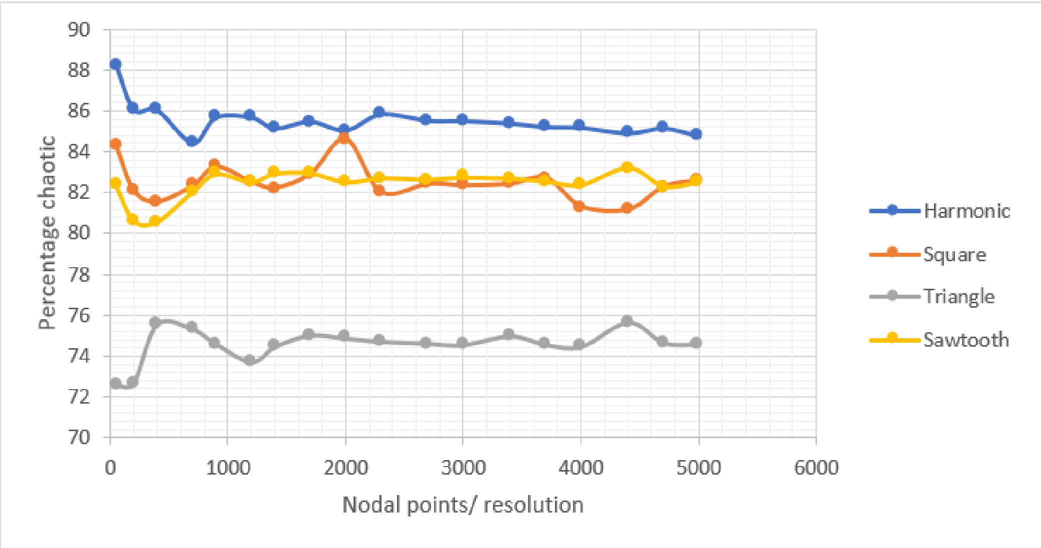

Figure 8: (d_i, v_i) = (-1,0) γ = 0.0168

Given above in figure 8 is the graphical representation of the average proportion of chaotic points given quantitively to be (85.7%, 82.5%, 74.4% and 82.4%) for harmonics, square, triangle and sawtooth wave respectively.

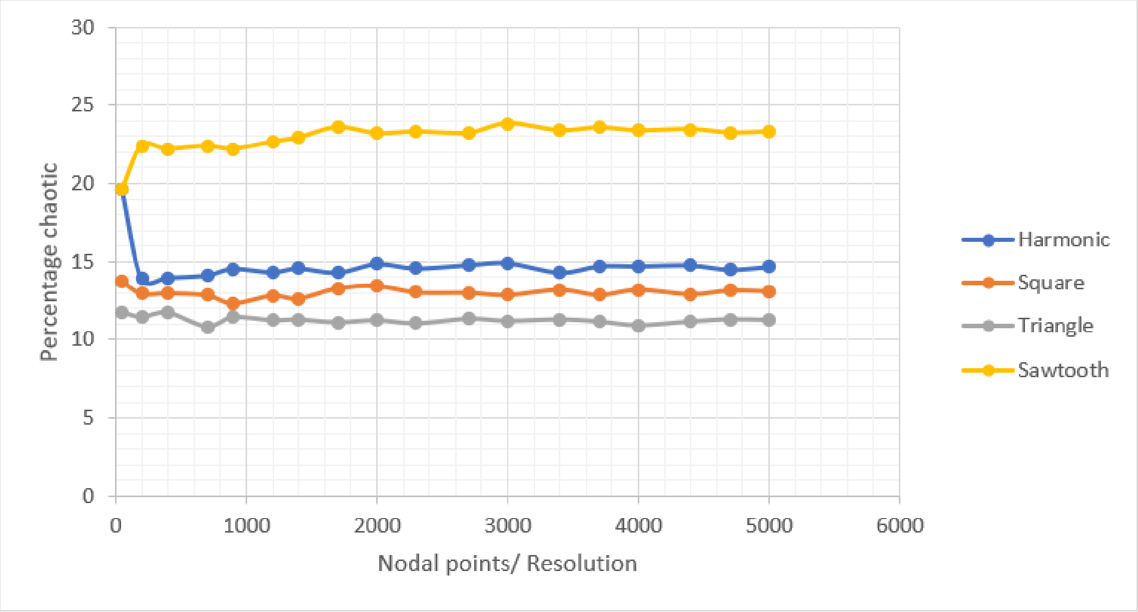

Figure 9: (d_i, v_i) = (0,0) γ = 0.168

Given above in figure 9 is the graphical representation of theaverage proportion of chaotic points given quantitively to be(14.8%, 13.1%, 11.3% and 22.7%) for harmonics, square, triangle and sawtooth wave respectively.

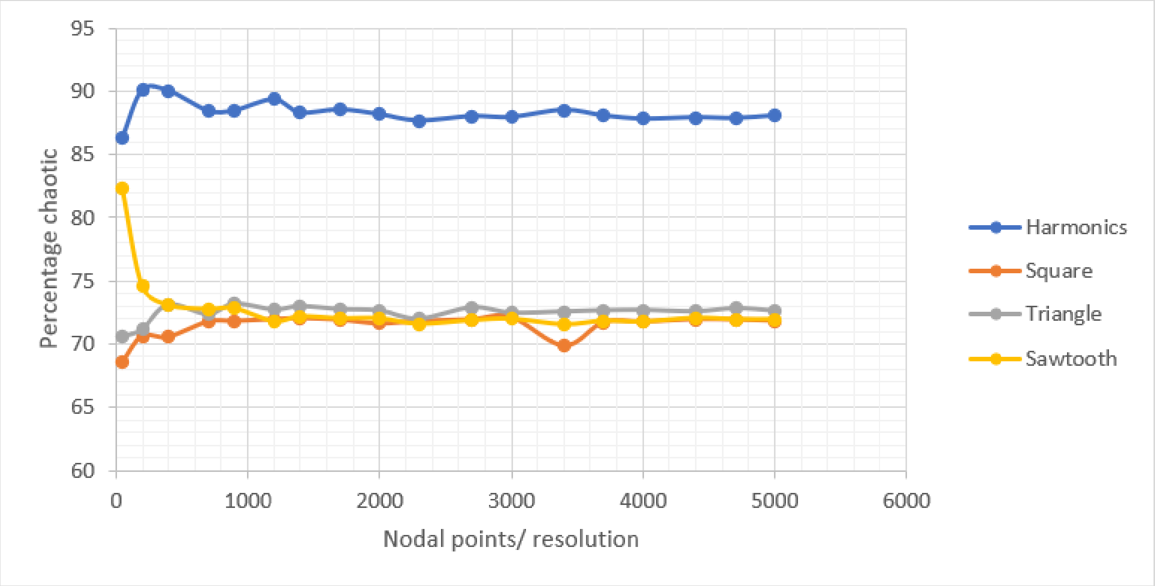

Figure 10: (d_i, v_i) = (-1,0) γ = 0.0168

Given above in figure 10 is the graphical representation of theaverage proportion of chaotic points given quantitively to be (88.4%, 71.5%, 72.5% and 72.8%) for harmonics, square, triangle and sawtooth wave respectively.

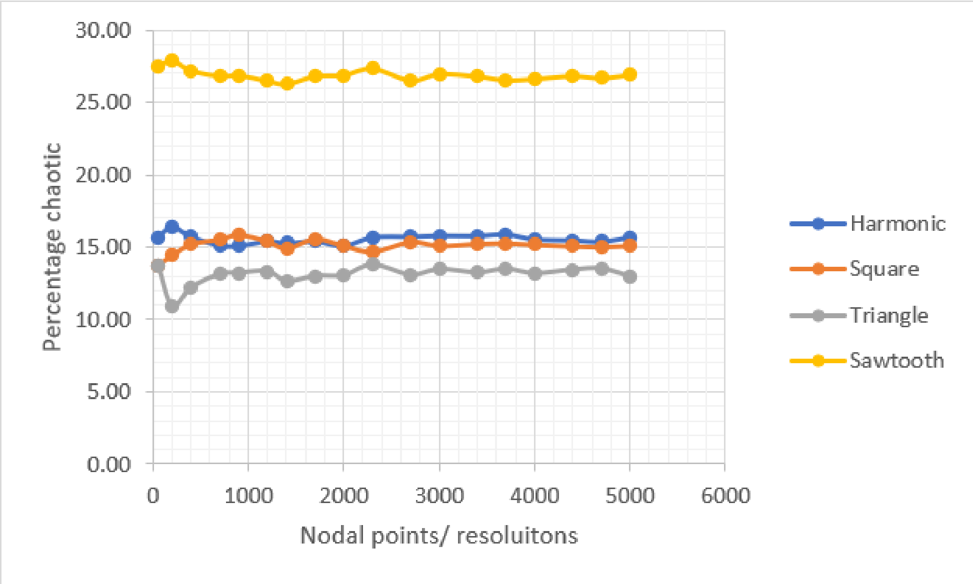

Figure 11: $(d_i, v_i) = (1,0)$ $\gamma = 0.168$.

Given above in figure 11 is the graphical representation of the average proportion of chaotic points given quantitively to be(15.6%, 15.2%, 13.1% and 28.9%) for harmonics, square, triangleand sawtooth wave respectively.

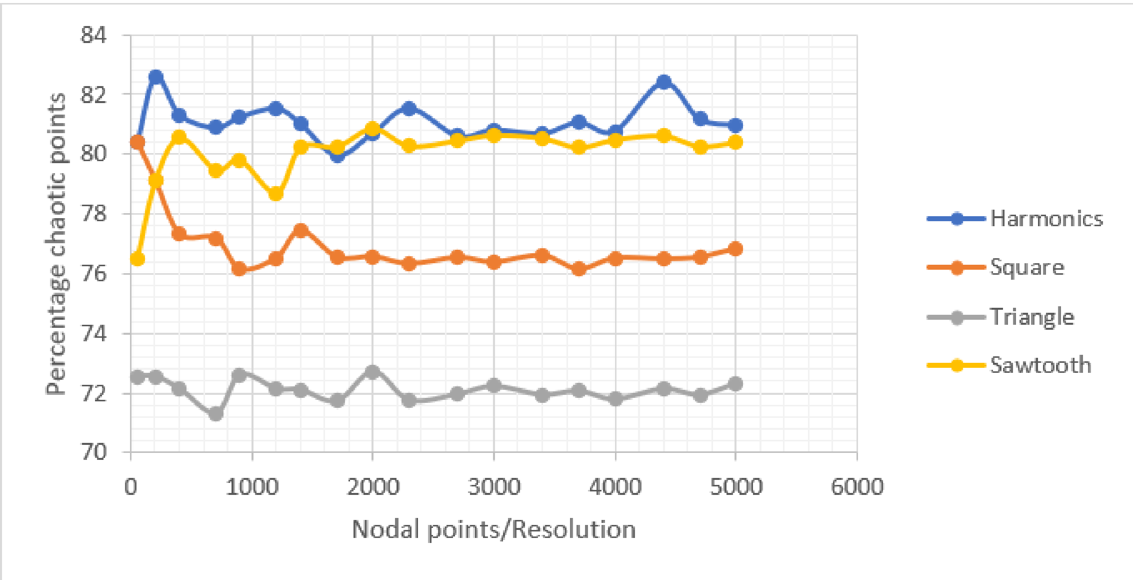

Figure 12: $(d_i, v_i) = (1,0)$ $\gamma = 0.0168$.

Given above in figure 12 is the graphical representation of the average proportion of chaotic points given quantitively to be (80.9%, 77.1%, 72.2% and 80.1%) for harmonics, square, triangle and sawtooth wave respectively.

V. Discussion:

From the simulated results generated above it has been observed that there is no qualitative difference in the percentage chaotic behaviour of a duffing oscillator under selected excitation regardless of the number of nodal points/ resolutions of interest within the amplitude range was observed, likewise no qualitative difference in the percentage chaotic points for the selected waveforms regardless of the equilibrium positions too.

Furthermore, in this research it was observed that the sawtooth wave will behave more chaotically compared to other waveforms at a higher damp coefficient.
Also, at lesser damping coefficient probability of parameter combinations to drive the duffing oscillator chaotically is higher but at a higher damping coefficient, the probability is lesser.

VI. Summary and conclusion:

The authors have been able to develop a python subroutine codes for the algorithms of Gram-schmidt orthogonalization in the determination of the average Lyapunov exponents and used the python subroutines to characterize the behaviour of the duffing oscillator (in the amplitude plane) under harmonic and selected periodic forcing.Thereafter, compared the results of the periodic forcing to the harmonic as a reference standard.

In this research journal the authors have discussed the behaviour of the duffing oscillator under three common periodic excitations (sawtooth, triangle and square wave). It was found in this research that duffing oscillator behaves distinctly dynamically when subjected to the various selected periodic excitations in reference to the reference system of harmonically excited duffing oscillator. The behaviour of the duffing oscillator when driven by the sawtooth wave external excitation is peculiar compared to other selected excitations at damping coefficient of 0.0168 as the sawtooth wave causes the duffing oscillator to have a higher percentage of points compared to other selected waves with respect to the harmonically excited duffing oscillator for same drive parameter value.

With further increase in the number of resolutions(nodal points of interest) examined along the plane of amplitude within the range of 0.07 to 1.5, it is interesting to note that there is no qualitative difference in the number of chaotic points regardless of the number of resolutions(nodal points of interest) selected within the plane of amplitude within the range of 0.07 to 1.5.

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