# An Optimal Analysis of Flow and Heat Transfer over a Slender Permeable Elastic Sheet with Variable Fluid Properties

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**Abstract:** The present article examines the influence of variable liquid properties on flow and heat transfer over a permeable stretching sheet with variable thickness. The transformed system of coupled non-linear ordinary differential equations is solved analytically via optimal homotopy analysis method (OHAM). Numerical results are analyzed graphically. Wall thickness parameter exhibits dual nature for flow and heat transfer patterns when it takes the value greater than 1 or less than. The skin friction and the wall temperature gradient are examined for influential parameters in this consideration.

Keywords: Variable thickness; Permeability; Skin friction; Nusselt number; OHAM.

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### I. INTRODUCTION

Fluid flows through permeable media are of enthusiasm for some fields of engineering and natural sciences, for example, oil recuperation, soil mechanics, material adsorption on solids, filtration, and polymer property estimations. In these designing and applied research fields, the flow through a permeable medium is normally treated by a basic relationship broadly referred to in a summed up frame as the Kozeny or Darcy law. In view of these application Abel et al. [1] examined the impact of permiablity on the flow and heat transfer of a non-Newtonian liquid over a non-isothermal stretching sheet. Pal and Mondal [2] applied Soret and Dufour effects, chemical reaction and thermal radiation on flow field over a porous stretching sheet. Stretching/shrinking porous sheet geometry is considered by Rosali et al. [3] and recorded the enhancement in the skin friction coefficient and the local Nusselt number for increasing permeability parameter. Recently, Bhatti et al. [4] continued the work of Ref. [3] by considering shrinking porous sheet. One of the important facts that all these researchers have concentrated on the pioneering work of Crane [5] and moreover numerous researchers have examined the nature of fluid flow by considering the geometry proposed by Crane[5](See Shehzad et al.[6], Prasad et al.[7], Vajravelu et al.[8], Hayat et al.[9], Zeeshan et al.[10])

All the above researchers have explored the nature of flow and heat transfer of a Newtonian/non Newtonian fluid by considering linear/ nonlinear stretching sheet. However, there is one more special type of nonlinear stretching of the sheet recorded in the literature, namely, nonlinear stretching with variable thickness. Here, the boundary conditions are different from conventional nonlinear stretching sheet problems such as  $u_w(x) = U_0(x+b)^m$  at  $y = A(x+b)^{(1-m)/2}$ . For all practical purposes deforming substances like needles and nozzles were the base for variable sheet thickness. A stretching sheet with a variable thickness can be more close to the situation in practical applications. In the year 2012 Fang et al. [11] coined the word 'variable thickness' and analysed the flow pattern numerically. Many researchers extended the work of Fang et al. [11] with the addition of heat and mass transfer (Khader and Megahed [12], Prasad. et al.[13], Salahuddin et al. [14], Prasad et al.[15-17]).

The object of present analysis is to predict the impact of variable liquid properties on the flow and heat transfer of fluid towards permeable stretching sheet of variable thickness. The relevant problems are formulated. Convergent series solutions of governing equations are constructed by optimal homotopy analysis method (OHAM) ([18]-[20]). Graphical results are used to elaborate the impacts of involved parameters.

# **II. MATHEMATICAL FORMULATION**

Consider a steady two-dimensional viscous incompressible fluid flow past a permeable stretching sheet with variable thickness. The origin is located at the slit, through which the sheet (see Fig. 1) is drawn in the fluid. The Flow caused due to nonlinear stretching sheet is restricted in domain y>0. Stretching velocity of the permeable sheet is  $U_w(x) = U_0 (x + b)^m$  where  $U_0$  and b are constants (m is the velocity exponent pa-

rameter). The problem statements in the absence of pressure gradient are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$
(1)
$$\rho_{\infty} \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left( \mu(T) \frac{\partial u}{\partial y} \right) - \frac{\mu(T)}{\rho_{\infty} K'} u,$$
(2)
$$\rho_{\infty} c_{p} \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left( k \left( T \right) \frac{\partial T}{\partial y} \right).$$
(3)





where, (u,v) are the fluid velocity components in the stream wise and cross-stream directions, respectively. The subscript denotes partial differentiation with respect to the independent variables  $\rho_{\infty}$  is the constant fluid density,  $\mu(T)$  is the coefficient of viscosity and K' is the permeability of the porous medium. Here in this work  $\mu(T)$  is considered to vary as an inverse function of temperature (see for details Prasad et al. [18]). The appropriate boundary conditions for the problem are

$$u(x, y) = U_w = U_0(x+b)^m, \ v(x, y) = 0, \ T(x, y) = T_w = \frac{C}{l}(x+b)^r \ at \quad y = A(x+b)^{1-m/2},$$

$$u(x, y) \to 0, \qquad T(x, y) \to T_{\infty} \ as \quad y \to \infty.$$
(4)

Let the dimensionless similarity variable be

$$\eta = y \sqrt{\frac{m+1}{2} \frac{U_0}{v_{\infty}}} (x+b)^{\frac{m-1}{2}},$$

$$\psi(x,y) = f(\eta) \sqrt{\frac{2}{m+1} U_0 v_{\infty}} (x+b)^{\frac{m+1}{2}}, \quad \theta(\eta) = (T-T_{\infty})/(T_w - T_{\infty})$$
(6)

(6)

where  $\psi(x, y)$  identically satisfies the continuity Eq. (1), the velocity components can be written as

$$u = U_{w}f'(\eta) \quad and \quad v = -\sqrt{v_{\infty}\frac{m+1}{2}U_{0}} (x+b)^{\frac{m-1}{2}} \left[f(\eta) + \eta f'(\eta)\left(\frac{m-1}{m+1}\right)\right].$$
(7)

Using above, Eqs. (2)- (3) and (4) reduces to

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$$\left(\frac{f''}{(1-\theta/\theta_r)}\right)' + ff'' - \frac{2m}{(m+1)} f'^2 - \frac{K}{(1-\theta/\theta_r)} f' = 0,$$
(8)

$$\left[\left(1+\varepsilon\theta\right)\theta'\right]' + \Pr\left(f\theta' - \frac{2r}{m+1}\theta f'\right) = 0,\tag{9}$$

and the corresponding boundary conditions are  $(m \neq 1)$ 

$$f(\alpha) = \alpha \frac{(1-m)}{(1+m)}, f'(\alpha) = 1, \ \theta(\alpha) = 1, \ \theta(\infty) = 0, \ f'(\infty) = 0.$$
(10)

The non-dimensional parameters, namely, variable thickness  $(\alpha)$ , the fluid viscosity parameter  $(\theta_r)$ , the porous parameter (K) and the Prandtl number (Pr) are defined as

$$\alpha = A_{\sqrt{\frac{m+1}{2}\frac{U_0}{\nu_{\infty}}}}, \theta_r = \frac{T_r - T_{\infty}}{T_w - T_{\infty}} = -\frac{1}{\gamma(T_w - T_{\infty})}, \quad K = \nu_{\infty}/K'b \quad , \Pr = \frac{\nu_{\infty}}{\alpha_{\infty}} \quad .$$

The value of  $\theta_r$  is determined by the viscosity of the fluid and  $\eta = \alpha = A \sqrt{\frac{m+1}{2} \frac{U_0}{v_{\infty}}}$  is the plate surface. In order to facilitate the computation, we define  $f(\xi) = f(\eta - \alpha) = f(\eta)$  and  $\theta(\xi) = \theta(\eta - \alpha) = \theta(\eta)$ . Now the equations become

$$\left(\frac{f''}{(1-\theta/\theta_r)}\right)' + ff'' - \frac{2m}{(m+1)}f'^2 - \frac{K}{(1-\theta/\theta_r)}f' = 0,$$
(11)

$$\left[\left(1+\varepsilon\theta\right)\theta'\right]' + \Pr\left(f\theta' - \frac{2r}{m+1}\theta f'\right) = 0,\tag{12}$$

and the corresponding boundary conditions are  $(m \neq 1)$ 

$$f(0) = \alpha \frac{(1-m)}{(1+m)}, f'(0) = 1, \theta(0) = 1, \theta(\infty) = 0, f'(\infty) = 0 ,$$
(13)

where the prime denotes the differentiation with respect to  $\xi$ . With reference to variable transformation, the integration domain will be fixed from 0 to  $\infty$ . The shear stress and the wall temperature gradient respectively become  $f''(\alpha) = f''(0)$  and  $\theta'(\alpha) = \theta'(0)$ . The values of engineering interest are the local skin friction  $C_{fx}$  and the local Nusselt number  $Nu_x$  defined as

$$C_{f_x} = \frac{2\nu_{\infty} \left(u_y\right)_{y=A(x+b)^{\frac{1-m}{2}}}}{U_w^2} = 2\sqrt{(m+1)/2} \left(\operatorname{Re}_x\right)^{-1/2} f''(0),$$

$$Nu_x = \frac{(x+b)(T_y)_{y=A(x+b)^{\frac{1-m}{2}}}}{(T_w - T_\infty)} = -\sqrt{(m+1)/2} \left(\operatorname{Re}_x\right)^{1/2} \theta'(0),$$
(14)

where  $\operatorname{Re}_{x} = U_{w}(x+b)/v_{\infty}$  is the local Reynolds number.

# **III. SEMI-ANALYTICAL NUMERICAL SOLUTION METHOD**

The governing equations are highly nonlinear, coupled ODEs with variable coefficients. We use the optimal homotopy analysis method (OHAM) to obtain appropriate analytic solutions to equations (11) and (12) with associated boundary conditions (13). The OHAM has been successfully applied to a wide variety of nonlinear problems (see [18-20]).

We choose the auxiliary linear operators as

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$$L_{f} = \frac{d^{3}}{d\xi^{3}} - \frac{d}{d\xi}, \quad L_{\theta} = \frac{d^{2}}{d\xi^{2}} - f, \text{ and } \quad L_{\phi} = \frac{d^{2}}{d\xi^{2}} - f.$$
(15)

Initial approximations satisfying the boundary conditions (13) are found to be

Let us consider the so-called zero-th order deformation equations

$$(1-q)L_{f}\left[\hat{f}(\xi;q) - f_{0}(\xi)\right] = qH_{f}(\xi)\hbar_{f}N_{f}\left[\hat{f}(\xi;q),\hat{f}(\xi;q)\right],$$

$$(1-q)L_{\theta}\left[\hat{\theta}(\xi;q) - \theta_{0}(\xi)\right] = qH_{\theta}(\xi)\hbar_{\theta}N_{\theta}\left[\hat{f}(\xi;q),\hat{\theta}(\xi;q)\right],$$

$$(17) \quad (1-q)L_{\phi}\left[\hat{\phi}(\xi;q) - \phi_{0}(\xi)\right] = qH_{\phi}(\xi)\hbar_{\phi}N_{\phi}\left[\hat{\theta}(\xi;q),\hat{\phi}(\xi;q)\right].$$

$$(18)$$

Here  $q \in [0,1]$  is an embedding parameter, while  $\hbar_f \neq 0$ ,  $\hbar_\theta \neq 0$  and  $\hbar_\phi \neq 0$  are the convergence control parameters. With these approximations, we may evaluate the residual error and minimize it over the parameters  $\hbar_f$ ,  $\hbar_\theta$  and  $\hbar_\phi$  in order to obtain the optimal value of  $\hbar_f$ ,  $\hbar_\theta$  and  $\hbar_\phi$  giving the least possible residual error. To do so, one may use the integral of squared residual errors, however this is very computationally demanding. To get around this, we use the averaged squared residual errors, defined by

$$\overline{\mathbf{E}_{n}^{f}}(\hbar_{f}) = \frac{1}{M+1} \sum_{k=0}^{M} \left( \mathbf{N}_{f} \left[ f_{[M]}(\xi_{k}), \theta_{[M]}(\xi_{k}) \right] \right)^{2},$$

$$\overline{\mathbf{E}_{n}^{\theta}}(\hbar_{\theta}) = \frac{1}{M+1} \sum_{k=0}^{M} \left( \mathbf{N}_{\theta} \left[ f_{[M]}(\xi_{k}), \theta_{[M]}(\xi_{k}), \phi_{[M]}(\xi_{k}) \right] \right)^{2},$$

$$\overline{\mathbf{E}_{n}^{\phi}}(\hbar_{\phi}) = \frac{1}{M+1} \sum_{k=0}^{M} \left( \mathbf{N}_{\phi} \left[ f_{[M]}(\xi_{k}), \theta_{[M]}(\xi_{k}), \phi_{[M]}(\xi_{k}) \right] \right)^{2},$$

where  $\xi_k = k / M$ , k = 0, 1, 2, ..., M. For different order approximations, the CPU time required for obtaining the approximate solutions will vary. Table 1 lists the values of individual average residual errors by considering the optimal values of  $\hbar_f = -1.20389$ ,  $\hbar_{\theta} = -1.021035$ ,  $\hbar_{\phi} = -1.026977$ , which have been obtained by minimizing the averaged residual errors at the 10<sup>th</sup> order approximation. Results are validated by comparing the present results with the results available in the literature (See Table.2).

#### **IV. RESULT AND DISCUSSION**

In order to understand the mathematical model, we present the numerical results graphically for the horizontal velocity profile f' and the temperature profile  $\theta$  with  $\xi$  for different values of  $\alpha$  and parameters m, K,  $\theta_r$ , Pr,  $\varepsilon$  and r in Figs. 2 to 6. The skin friction f''(0) and the wall temperature gradient  $\theta'(0)$  are tabulated in Table 3.

Fig. 2 (a) and Fig. 2 (b) illustrates the effect of m on f' for increasing values of  $\alpha$ . It shows that the velocity decreases with an increase in the value of m. This implies that the momentum boundary layer thickness becomes thinner as m increases. Fig. 3(a) portrays the velocity distribution for different values of K. It indicates that the porous parameter opposes the transport phenomena. This is due to the fact that the variation in K leads to the variation of the Lorentz force which in turn produces more resistance to the transport phenomena. It is clearly seen from the graph that the momentum boundary layer thickness decreases as K increases, and hence there is an increase (in absolute sense) in the velocity gradient f''(0) at the surface (see Table 3). Fig. 3(b) shows the effect of  $\theta_r$  on f'. It is observed that the velocity distribution asymptotically tends to zero. This is due to the fact that for a given fluid, when  $\theta_r$  is smaller, higher is the temperature difference between the wall and the ambient fluid. The results clearly reveal that  $\theta_r$  is the indicator of the variable viscosity with temperature which has a substantial effect on the velocity component

f' and hence on the skin friction.

We shall now turn our attention to the influence of various parameters on the temperature field  $\theta$ . The effect of m on  $\theta$  is exhibited in Fig. 4(a) and Fig. 4(b). The effect of increasing values of m is to increase the temperature distribution. This is in conformation with the fact that an increase in m leads to an increase in the thermal boundary layer thickness which is also true even for non-zero values of K. As explained above, increase in K increases the temperature as shown in Fig. 5(a). Moreover, the rate of cooling of the end product is important in several manufacturing processes such as metal and polymer extrusion, which will confirm the quality of the end product. Fig. 5(b) explains effect of  $\theta_r$  on  $\theta$ . From the graphical representation it is seen that the effect of increasing value of  $\theta_r$  is to enhance the temperature. That is, an increase in  $\theta_r$  results in an increase in the thermal boundary layer thickness. The effect of Pr on  $\theta$  can be found from Fig. 6(a). The figure demonstrates that an increase in Pr (means decrease in the thermal conductivity  $k_{\infty}$ ) leads to a decrease in the temperature. Hence the thermal boundary layer thickness decreases as Pr increases. This is because fluid with a higher values of Pr posses a large heat capacity and hence intensifies the heat transfer. Therefore, cooling of the heated sheet can be improved by choosing a coolant with a large Pr. Fig. 6(b) displays the effects of  $\varepsilon$  on  $\theta$ . Fluid temperature is found to increase with increasing values of  $\varepsilon$  which leads to a fall in the rate of heat transfer. That is, the assumption of temperature dependent thermal conductivity suggests a reduction in the magnitude of the transverse velocity by a quantity  $\partial k(T)/\partial y$  which can be seen in Eq. (2.3). Therefore, the rate of cooling is much faster for the coolant material having small thermal conductivity parameter. Fig. 6(c) elucidates the effect of r on  $\theta$  in the boundary layer. An increase in the value of r leads to decrease in  $\theta$  and this is because when r > 0, heat flows from the stretching sheet into the ambient medium and, when r < 0, the temperature gradient is positive and heat flows into the stretching sheet from the ambient medium.

An interesting observation from the above results is that the velocity and temperature distributions depend heavily on the parameters m and  $\alpha$ . It is noticed that the velocity at any point near the plate decreases monotonically as  $\alpha$  increases for m < 1. Also it is obvious from the figures that the thickness of the boundary layer becomes thinner for higher values of  $\alpha$  when m < 1, but the reverse is true for  $m \ge 1$  (see Fig. 2(a) and Fig. 2(b)). This is due to the induced mass transfer. This momentum transfer accelerates the fluid particle at the downward region. This kind of significant change in the velocity can also be seen for positive values of m, the stretching sheet case. For higher values of  $\alpha$ , thermal boundary layer becomes thinner for  $m \ge 1$  (see Fig. 4(a) and Fig. 4(b)).

The effects of the physical parameters on the skin friction f''(0) and the Nusselt number  $\theta'(0)$  are presented in Table 3. It is noticed that the effect of increasing values of the parameters m, K,  $\theta_r$ ,  $\varepsilon$  is to decrease f''(0) and to increase  $\theta'(0)$ . The effect of increasing values of r and the Pr is to decrease  $\theta'(0)$ . Further it is observed that an increase in  $\alpha$  leads to a decrease in f''(0) as well as  $\theta'(0)$  for m < 1, whereas an opposite trend is observed as m > 1.

# V. CONCLUSIONS

The important findings are as follows:

- The dimensionless velocity and temperature distributions at any point near the plate decrease when m < 1 and the thickness of the boundary layer becomes thinner when m < 1 and a reverse is true for  $m \ge 1$ .
- In the presence of temperature dependent thermo-physical properties, the effect of increasing porous parameter is to decrease the velocity field. However, quite opposite is true with the thermal boundary layer.
- > The non-dimensional heat transfer rate reduces for increasing m.
- The effect of the Pr is to decrease the thermal boundary layer thickness and the wall temperature gradient.
- > The effect of  $\mathcal{E}$  is to enhance the temperature in the flow region and is reversed in the case of the r.
- Of all the parameters, the variable thermo-physical property parameters have strong effects on the drag, heat transfer characteristics, the horizontal velocity field and the temperature field.

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Table 1: Individual average residual error as a function of the number of iterations. CPU time requiredto calculate the solution is also listed. Parameter values are fixed atPr = 1.09,

Le = 1	2.0, <i>ε</i>	$\epsilon_1 = \epsilon_2 = 0.1, \ \theta_r \to 0$	$\infty, \alpha = \frac{1}{2}, m = -\frac{1}{3},$	K = 0. We have optimized the second secon	mal convergence control pa-
		rameter values of	$\hbar_f = -1.20389, \ \hbar_f$	$h_{g} = -1.021035, \ \hbar_{\phi} =$	= -1.026977.
	10	$\overline{f}$			

n	$\overline{\mathcal{E}_n^f}$	$\overline{\mathcal{E}_n^{ heta}}$	$\overline{\mathcal{E}^{\phi}_n}$	CPU time (Sec)
2	5.86×10 <sup>-3</sup>	1.45×10 <sup>-2</sup>	7.59×10 <sup>-2</sup>	3.70
4	2.31×10 <sup>-3</sup>	1.96×10 <sup>-3</sup>	2.03×10 <sup>-2</sup>	11.05
6	5.14×10 <sup>-4</sup>	4.32×10 <sup>-4</sup>	$1.74 \times 10^{-2}$	28.83
8	4.86×10 <sup>-5</sup>	7.19×10 <sup>-5</sup>	5.22×10 <sup>-3</sup>	78.01
10	2.12×10 <sup>-6</sup>	6.43×10 <sup>-6</sup>	4.09×10 <sup>-4</sup>	149.96
12	1.73×10 <sup>-6</sup>	7.95×10 <sup>-6</sup>	$4.82 \times 10^{-4}$	259.53
14	3.21×10 <sup>-7</sup>	4.62×10 <sup>-6</sup>	2.62×10 <sup>-4</sup>	496.31
16	4.67×10 <sup>-8</sup>	1.03×10 <sup>-6</sup>	2.85×10 <sup>-5</sup>	561.59
18	2.05×10 <sup>-7</sup>	5.95×10 <sup>-7</sup>	2.68×10 <sup>-5</sup>	766.29
20	1.66×10 <sup>-7</sup>	5.07×10 <sup>-7-</sup>	2.07×10 <sup>-5</sup>	961.20

**Table 2.** Comparison of skin friction -f''(0) for different values of m and  $\alpha$  with  $\theta_r \to \infty$ ,  $\varepsilon = K = 0.0$ .

	-f''(0)						
m	$\alpha = \frac{1}{2}$			$\alpha = \frac{1}{4}$			
	Fang et al.[11]	Khader and Me- gahed [12] when $\lambda = 0.0$	Present Work by OHAM	Fang et al.[11]	Khader and Megahed [12] when $\lambda = 0.0$	Present Work OHAM	by
10.0	1.0603	1.0603	1.06032	1.1433	1.1433	1.14331	
5.0	1.0486	1.0486	1.04866	1.1186	1.1186	1.11862	
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
0.0	0.9576	0.9577	0.95773	0.7843	0.7843	0.78441	
-1/3	1.0000	1.0000	1.0000	0.5000	0.5000	0.5000	
-1/2	1.1667	1.1666	1.16659	0.0833	0.8322	0.83221	

Table 3. Variation of skin friction and wall-temperature gradient for different values of physical parameters.

								1		1	
Dr	£	r	θ	V	т	$\alpha = 0.0$	0	$\alpha = \frac{1}{10}$		$\alpha = \frac{1}{4}$	
II.	Ū	7	0 <sub>r</sub>	Λ		<i>f</i> "(0)	$\theta'(0)$	<i>f</i> "(0)	$\theta'(0)$	<i>f</i> "(0)	$\theta'(0)$
					-1/	0.580	-1.819	0.55434	-1.982	-0.22406	-2.08570
					2	523	807	0	426	0	6
					-1/	-0.040	-1.647	-0.2133	-1.686	-0.60039	-1.75893
					3	846	321	54	796	2	4
					0.	0.732	-1.243	-0.8100	-1.282	-0.93467	-1.33396
1.0	0.1	1.0	-5.	0.0	0	229	645	50	226	4	1
1.0	0.1	1.0	0	0.0	1.	-1.178	-0.881	-1.1786	-0.881	-1.17860	-0.88174
					0	604	745	04	745	4	5
					5.	-1.410	-0.881	-1.3687	-0.599	-1.30872	-0.55414
					0	420	745	23	053	4	4
					10	-1.458	-0.572	-1.4078	-0.532	-1.33547	-0.47579
					.0	498	057	02	096	3	9
					-1/	0.673	-1.901	0.44509	-2.033	-0.45748	-2.11503
					2	353	353	4	648	1	6
					-1/	-0.264	-1.585	-0.4807	-1.632	-0.81915	-1.71483
					3	173	368	40	266	2	5
					0.	-0.888	-1.214	-0.9649	-1.247	-1.08382	-1.29976
1.0	0.1	1.0	-5.	0.2	0	657	101	20	645	6	0
1.0	0.1	1.0	0	0.2	1.	-1.290	-0.858	-1.2909	-0.858	-1.28943	-0.85129
					0	938	403	37	401	7	3
					5.	-1.507	-0.614	-1.4648	-0.583	-1.40290	-0.52180
					0	071	852	36	143	0	7
					10	-1.552	-0.558	-1.5009	-0.518	-1.42651	-0.44233
					.0	348	263	69	078	8	1
				0.0		-1.350	-0.570	-1.3066	-0.529	-1.26423	-0.48926
1.0				0.0		809	495	84	089	9	3
	0.1	1.0	-1	0.5	10	-1.559	-0.534	-1.5142	-0.491	-1.47046	-0.45151
	0.1	1.0	0.0	0.0	.0	542	179	79	750	8	0
				1.0		-1.742	-0.515	-1.6967	-0.480	-1.75150	-0.38977
				1.0		674	136	65	624	2	2
			-1		5	-1.397	-0.626	-1.3605	-0.594	-1.32490	-0.56411
1.0	0.1	1.0	00	0.2	0	307	948	70	903	1	3
			-5.		Ŭ	-1.507	-0.614	-1.4648	-0.583	-1.42388	-0.55271

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			0			071	852	36	143	1	9
			-2.			-1.798	-0.601	-1.7390	-0.569	-1.68209	-0.54086
			0			244	746	80	686	5	8
		2.0				-1.552	-0.355	-1.5011	-0.309	-1.45152	-0.27021
		-2.0				788	966	37	924	4	6
		1.0				-1.552	-0.425	-1.5010	-0.381	-1.45156	-034383
		-1.0				633	808	77	940	4	9
1.0	0.1	0.0	-5.	0.2	10	-1.552	-0.493	-1.5010	-0.451	-1.45160	-0.41459
1.0	0.1	0.0	0	0.2	.0	486	188	21	264	4	5
		1.0				-1.552	-0.558	-1.5009	-0.518	-1.45164	-0.48268
		1.0				348	263	69	078	3	4
		2.0				-1.552	-0.6211	-1.5009	-0.582	-1.45168	-0.54829
		2.0				217	79	21	551	2	3
	0.0					-1.552	-0.615	-1.5008	-0.568	-1.45146	-0.52346
	0.0			0.2	10	152	189	81	135	1	0
	0.1	1	-5.			-1.552	-0.558	-1.5009	-0.518	-1.45146	-0.47991
1.0	0.1					349	262	69	077	5	9
1.0	0.2	1.0	0	0.2	.0	-1.552	-0.514	-1.5010	-0.479	-1.45147	-0.44590
	0.2					525	121	52	130	5	2
	03					-1.552	-0.478	-1.5011	-0.447	-1.45149	-0.41847
	0.5					683	737	30	831	0	9
0.7						-1.552	-0.463	-1.5012	-0.438	-1.45189	-0.41892
1						851	371	99	500	6	2
1.0						-1.552	-0.558	-1.5010	-0.519	-1.45185	-0.48567
1.0					347	264	54	363	0	4	
2.0						-1.550	-0.863	-1.5004	-0.776	-1.45175	-0.69477
2.0	0.1	1.0	-5.	0.2	10	947	580	00	217	7	8
5.0	0.1	1.0	0	0.2	.0	-1.549	-1.534	-1.4997	-1.866	-1.45191	-1.11338
5.0						337	385	75	861	7	9
7.0						-1.549	-1.874	-1.4997	-1.569	-1.45214	-1.29275
2						028	454	81	589	5	1
10.						-1.548	-2.296	-1.4999	-1.866	-1.45248	-1.48449
0						910	484	42	861	4	3

	An	Optimal	Analysis	of Flow a	and Heat Tro	nsfer over	· a Slender	Permeable	Elastic Sh	eet with Variable
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