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Modified Kaptur's Measures of Entropy and Directed Divergence on Imposition of Inequality Constraints on Probabilities

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Abstract: - In this paper, identifying the infirmities of Kapur's [1] measures of entropy, we have studied measures of entropy, which are revised from Kapur's [1] measures of entropy. With the improved entropy containing better rang of validity, we also have obtained globally measures of entropy.

Keywords: - Directed Divergence, Symmetric Directed Divergence, Measures of Inaccuracy, Measures of Information Improvement, Concave, Globally Measures of Entropy, Generlised Information Improvement.

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I. INTRODUCTION

The problem of distribution of maximum entropy was solved by Freund and Saxena [3] not merely with maximization of Shannon's [2] but also in the unavailability of moment constraints and determined the algorithm for obtaining the *MAX ENT* distribution. Kapur [1] provides the event conditional entropy, cross entropy also defined the cross entropy measures and are used to solve a number of entropy, cross entropy minimization. The cross entropy maximization problems incorporating inequality constraints on probabilities and maximum entropy probabilities distribution having inequality constraints on probabilities. The objective of the present paper is to examine some modified Kapur's [1] measures of entropy when inequality constraints are imposed on probabilities; and we have specified the infirmities of Kapur's [1] measures of entropy and then studied the measures of entropy revised from Kapur's [1] measures of entropy. The globally measures of entropy, measures of directed divergence ,measures of inaccuracy, symmetric directed divergence, measures of information improvement, generlised information improvement are acquired resembling to the new measures of entropy.

In section 2, some explanatory considerations are introduced containing the analysis on the basic concepts definitions and properties of measures of entropy and also proposed new revised measures of entropy. In section 3, we examine the various properties of measures of entropy that are defined in section 2. In section 4, we derive measures of directed divergence, measures of symmetric directed divergence, measures of inaccuracy, measures of information improvement and measures of generlised information improvement corresponding to the new revised measure of entropy**.** In section 5, conclusive comments on the present paper are discussed. The references of the paper are cited in section 6.

II. PRELIMINARIES

The moment constraints,

$\sum_{i=1}^{n} p_i g_r(x_i) = a_r$ where r = 1,2,….., m (2.1)

and the inequality constraints $0 \leq a_i \leq p_i \leq b_i \leq 1$, where i= 1,2,…..,n (2.2)

The inequality constraints (2.2) are answered i.e. each proportion or probability is constrained to exist between particular limit. If we maximize Shannon's [2] measure of entropy conditional to these moment, the probabilities obtained may not answer inequality constraints (2.2). Hence, we get a complicated mathematical programming wherein the derived inequality constraints needs to be fixed precisely. The exclusive convenient answer hitherto is the only moment constrain i.e. the natural constraint.

The following inequality constraint are recommended on probabilities $p_1, p_2, \ldots, p_n$ of a probability distribution meeting the natural constraints.

Let P = $(p_1, p_2, \ldots\ldots, p_n)$, $\sum_{i=1}^{n} p_i$ =1, $p_1 \geq 0, \ldots\ldots p_n \geq 0$, (2.3)

There may be an infinite of probability distribution satisfying (2.1), (2.2), and (2.3) by maximizing Shannon's [2] measure of entropy

$$S(P) = -\sum_{i=1}^{n} p_i ln p_i \quad (2.4)$$

We appropriate will make the case when the inequality constraints are $p_1 \geq 0, p_2 \geq 0, \ldots\ldots p_n \geq 0$ but is not an appropriate measure under constraint (2.2),

Kapur [1] has studied the following measures of inverse maximum of entropy principle to satisfy this condition stated above.

$$K_1(P) = -\sum_{i=1}^{n}(p_i - a_i)\ln(p_i - a_i) - \sum_{i=1}^{n}(b_i - p_i)\ln(b_i - p_i) \quad (2.5)$$

$$K_2(P) = \frac{1}{1-\propto}\left[\sum_{i=1}^{n}(p_i - a_i)^{\propto} - 1 + \sum_{i=1}^{n}(b_i - p_i)^{\propto} - 1\right] \quad , \propto \neq 1 \quad (2.6)$$

$K_1(P)$ has an infirmity excluding any parameter.

$K_2(P)$ contains a parameter $\propto$ but it is not defined for $\propto = 1$, when $\propto < 0$

$K_2(P)$ ceasing to be concave function of p_i, which an essential condition for measure of entropy.

For any probability distribution, P = $(p_1, p_2, \ldots\ldots, p_n)$ a measure of entropy need to satisfy for the following conditions;

i) It should be a continuous function of $p_1, p_2, \ldots\ldots, p_n$ since if $p_1, p_2, \ldots\ldots, p_n$ change by small amounts afterward their maximum value changes by a small amounts.
ii) It should be permutationally symmetric function of $p_1, p_2, \ldots\ldots, p_n$ since if $p_1, p_2, \ldots\ldots, p_n$ are permuted among themselves, then their maximum value remain same.
iii) Since, maximum it mean that minimizing the value p_{max}, it should be maximum subject to $\sum_{i=1}^{n} p_i = 1$ is attended when $p_1 = p_2 = \ldots = p_n = 1/n$ and the maximum value is an increasing function of n.
iv) It should be always be non negative and vanishes when, $p_i = q_i$ for all i = 1,….,n.
v) Its minimum value should be zero for each n degenerate distribution P = (0…,1….,0),1 at i^{th} place and 0 at remaining place, $1 \leq i \leq n$.

In this chapter, we have indicated the infirmities of Kapur's [1] measures of entropy and then studied the measures of entropy surpassing these infirmities. Thus, we define following measures of entropy

$$V_{\propto}(P) = \frac{\sum_{i=1}^{n}(p_i - a_i)^{\propto} + \sum_{i=1}^{n}(b_i - p_i)^{\propto} - \sum_{i=1}^{n}(b_i - a_i)}{1-\propto} \qquad \propto > 0, \propto \neq 1 \quad (2.7)$$

$$E[V_{\propto}(P)] = \frac{\sum_{i=1}^{n}(p_i - a_i)^{\propto} + \sum_{i=1}^{n}(b_i - p_i)^{\propto} - \sum_{i=1}^{n}(b_i - a_i)}{\propto - 1} \qquad \propto < 0, \propto \neq 1 \quad (2.8)$$

$K_2(P) \to \infty$ as $\propto \to 1$ (2.9)

$$\lim_{\propto \to 1} V_{\propto}(P) = \frac{\lim_{\propto \to 1}\left[\sum_{i=1}^{n}(p_i - a_i)^{\propto} + \sum_{i=1}^{n}(b_i - p_i)^{\propto} - \sum_{i=1}^{n}(b_i - a_i)\right]}{1-\propto}$$

$$= -\sum_{i=1}^{n}(p_i - a_i)\ln(p_i - a_i) - \sum_{i=1}^{n}(b_i - p_i)\ln(b_i - p_i)$$

$= K_1(P)$

$$\lim_{\propto \to 1} V_{\propto}(P) = K_1(P) \quad (2.10)$$

III. PROPERTIES OF $V_{\propto}(P)$ AND E$[V_{\propto}(P)]$

The measures $V_{\propto}(P)$ and E$[V_{\propto}(P)]$ have the following properties:

i. We define 0ln0 =0, $V_{\propto}(P)$ and E$[V_{\propto}(P)]$ are continuous functions of $p_1, p_2, \ldots p_n$ it changes by a small amount when $p_1, p_2, \ldots\ldots, p_n$ changes by small amounts
ii. Maximum value of $V_{\propto}(P)$ and E$[V_{\propto}(P)]$ are is remain same even if $p_1, \ldots\ldots, p_n$ are permutationally symmetric means if the n triplets (a_i, p_i, b_i) ,(i=1,2,…..n) are permutated amongst themselves, the measures does not change.
iii. $V_{\propto}(P)$ and E$[V_{\propto}(P)]$ are maximum subject to $\sum_{i=1}^{n} p_i = 1$ is attended when p_1= ------ = $p_n = \frac{1}{n}$ and the maximum value is an increasing function of n.

iv. Minimum value of $V_\propto(P)$ and E$[V_\propto(P)]$ are zero for the n degenerate distribution, where $1\leq i \leq n$.

v. $V_\propto(P)$ and E$[V_\propto(P)]$ are always non-negative

vi. Concavity of $V_\propto(P)$

$$\frac{\partial V_\propto(P)}{\partial p_i} = \frac{1}{1-\propto}\left[\sum_{i=1}^{n} \propto (p_i - a_i)^{\propto-1} + \sum_{i=1}^{n} \propto (b_i - p_i)^{\propto-1}\ (-1)\right],\ \alpha > 0$$

$$= \frac{1}{1-\propto}\left[\sum_{i=1}^{n} \propto (p_i - a_i)^{\propto-1} - \sum_{i=1}^{n} \alpha (b_i - p_i)^{\propto-1}\right] \quad (3.1)$$

$$\frac{\partial^2 V_\propto(P)}{\partial p_i^2} = -\propto\left[\sum_{i=1}^{n}(p_i - a_i)^{\propto-2} + \sum_{i-1}^{n}(b_i - p_i)^{\propto-2}\right] < 0 \quad \text{for} \quad \alpha > 0 \quad (3.2)$$

Since, the terms inside bracket are positive

$$\frac{\partial^2 V_\propto(P)}{\partial p_j \partial p_j} = 0\,,\ \ j \neq 1 \quad (3.3)$$

$V_\propto(P)$ is firmly concave function of $p_1, p_2, \ldots.. p_n$ so when $V_\propto(P)$ is maximized subject to natural linear constraints in (2.1) and (2.2), its local maximum will be its global maximum

(vii) Concavity property of $E[V_\propto(P)]$

$$\frac{\partial}{\partial p_i} E[V_\propto(P)] = \frac{\propto}{\propto-1}\left\{\sum_{i=1}^{n}(p_i - a_i)^{\propto-1} + \sum_{i=1}^{n}(b_i - p_i)^{\propto-1}(-1) + 0\right\} \quad \text{for} \propto < 0 \quad (3.4)$$

$$\frac{\partial^2}{\partial p_i{}^2} E[V_\propto(P)] = \frac{\propto(\propto-1)}{\propto-1}\left\{\sum_{i=1}^{n}(p_i - a_i)^{\propto-2} + \sum_{i=1}^{n}(b_i - p_i)^{\propto-2}\right\} < 0 \quad \text{for} \propto < 0 \quad (3.5)$$

$$\frac{\partial E[V_\propto(P)]}{\partial p_j \partial p_i} = 0,\ \ j \neq i \quad (3.6)$$

$E[V_\propto(P)]$ is also strictly concave function of $p_1, p_2, \ldots.. p_n$,so, when $E[V_\propto(P)]$ is maximized conditional to linear constraints (2.1) and (2.2) its local maximum will be its global maximum.

(viii) We already have proved that

$$\lim_{\propto\to 1} V_\propto(P) = K_1(P)$$

$$\lim_{\propto\to 1} V_\propto(P) = -\sum_{i=1}^{n}(p_i - a_i)\ln(p_i - a_i) - (b_i - p_i)\ln(b_i - p_i) \quad (3.7)$$

(ix) If each $a_{i=0}$ and $b_{i=1}$

It is natural constraints $0 \leq p_i \leq 1$

$$V_\propto(P) = \frac{\sum_{i=1}^{n} p_i^{\propto} + \sum_{i=1}^{n}(1 - p_i)^{\propto} - n}{1-\propto}$$

$$\lim_{\propto\to 1} V_\propto(P) = \lim_{\propto\to 1} \frac{\sum_{i=1}^{n} p_i^{\propto} + \sum_{i=1}^{n}(1 - p_i)^{\propto} - n}{1-\propto} \quad (3.8)$$

$$\lim_{\propto\to 1} V_\propto(P) = -\sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n}(1 - p_i)\ln(1 - p_i) \quad (3.9)$$

Which is Fermi Dirac [5] Entropy

(x) Additive Property

The measure is neither additive, nor sub-additive nor recursive but these can be done away with as they are not essential for our maximization problem.

(xii) Maximum value of $V_\propto(P)$

Now we will discuss about max value of $V_\propto(P)$ when it is maximized subject to natural constraints (2.3) and linear moment constraints in (2.1), by Lagrange's method we have

$$\propto (p_i - a_i)^{\propto-1} + \lambda + \propto (b_i - p_i)^{\propto-1} = 0 \quad (3.10)$$

$$p_i - a_i = k \quad (3.11)$$

$$\frac{b_i - p_i}{p_i - a_i} = v \quad (3.12)$$

k and v are positive constraints

$p_i = 1/n$

$k = \frac{1}{n} - a_i$

$= \frac{1 - na_i}{n}$

As $A = \sum_{i=1}^{n} a_i = na_i$

k = $\frac{1-A}{n}$ (3.13)

v = $\dfrac{\frac{nb_i-1}{n}}{\frac{1-na_i}{n}}$

Taking $B = \sum_{i=1}^{n} b_i = nb_i \; and \; A = \sum_{i=1}^{n} a_i = na_i$

v = $\frac{B-1}{1-A}$ (3.14)

On Solving (3.13)

v = $\frac{b_i-p_i}{p_i-a_i}$

$p_i \; = \frac{b_i+va_i}{v+1}$

$p_i \; = \frac{b_i-a_i+Ba_i-Ab_i}{B-A}$

it can be express as

$p_i = \frac{b_i-a_i+a_i\sum_{i=1}^{n} b_i-b_i\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i-\sum_{i=1}^{n} a_i}$ i = 1,2,…….,n (3.15)

The equations (3.12) and (3.15) gives a generalized uniform distribution, and it incorporates the usual uniform distribution when each $a_i = 0$ and each $b_i = 1$ then the equations (3.12) and (3.15) becomes

v = (n-1), $p_i = \frac{1}{n}$, i=1, 2,……, n

By modified Laplace's principal

"If we have no other clue about the probability distribution except should be equal $\sum_{i=1}^{n} p_i = 1$ and $a_i \le p_i \le b_i$, where a ≥ 0, b ≤ 1 then all the probabilities should be taken as equal".

Here, all the information about $p_1, p_2, \dots, p_n$ is permutationally symmetric in $p_1, p_2, \dots, p_n$ therefore, we choose uniform distribution as $p_1 = p_2 = \dots = p_n = \frac{1}{n}$

(xiii) Now the Maximum value of $V_\propto(P)$ is

$$V_\propto(P) = n\left(\frac{k^\propto + k^\propto v^\propto - k(1+v)}{1-\propto}\right)$$

$$= \frac{n^{1-\propto}\{(1-A)^\propto+(B-1)^\propto\}-(1-A)(B-A)}{1-\propto}$$

For $\propto = 1$

$$[V_\propto(P)]_{\propto=1} = \lim_{\propto\to 1} \frac{n^{1-\propto}\{(1-A)^\propto ln(1-A)+(B-1)^\propto ln(B-1)\} + \{(1-A)^\propto+(B-1)^\propto\} n^{1-\propto} \ln n(-1)}{-1}$$

$[V_\propto(P)]_{\propto=1}$ = - [(B-1) ln (B-1) + (1-A) ln (1-A) - (B-1)(1-A) ln n] (3.16)

IV. DIRECTED DIVERGENCE AND SYMMETRIC DIRECTED DIVERGENCE

4.1 Measures Of Directed Divergence

If P = (p_1, p_2-----,p_n) and Q = (q_1, q_2------,q_n) are two probability distributions with constraints

$p_i \ge 0, \quad (i = 1,2, \dots, n), \quad \sum_{i=1}^{n} p_i = 1 \quad, a_i \le p_i \le b_i$

$q_i \ge 0, \quad (i = 1,2, \dots, n), \quad \sum_{i=1}^{n} q_i = 1 \quad, a_i \le q_i \le b_i$ (4.1.1)

Then the measures of directed divergence of P from Q is a function of D (P; Q) must be suitable to following conditions:

a) D(P :Q) ≥ 0
b) D(P :Q) = 0 if and only if P = Q
c) D(P:Q) is a convex function of both p_1, p_2-----,p_n and q_1, q_2------,q_n

A measure is a convex function of both P and Q is found from Csiszer's [6] measure

D(P: Q) = $\sum_{i=1}^{n} q_i \phi\left(\frac{p_i}{q_i}\right)$ (4.1.2)

the prominent Kullback-Leibler [7] derived measure of directed divergence D_1(P: Q) of P from Q, corresponding to Shannon [2] measure of entropy as

Where

$D_1(P:Q) = \sum_{i=1}^{n} p_i \ln\frac{p_i}{q_i}$

Then the directed divergence corresponding to $V_\propto(P)$

$$D[V_\propto(P:Q)] = \frac{\sum_{i=1}^{n}(p_i-a_i)^\propto(q_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-p_i)^\propto(b_i-q_i)^{1-\propto}-\sum_{i=1}^{n}(b_i-a_i)}{\propto-1} \quad \alpha > 0 \qquad (4.1.3)$$

$$D\{E[V_\propto(P:Q)]\} = \frac{\sum_{i=1}^{n}(p_i-a_i)^\propto(q_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-p_i)^\propto(b_i-q_i)^{1-\propto}-\sum(b_i-a_i)}{1-\propto} \quad \alpha < 0 \qquad (4.1.4)$$

The obtained measures of directed divergence D(P:Q) meeting the following properties

(i) $D[V_\propto(P:Q)] = 0 \quad \Longrightarrow P = Q$ (4.1.5)

(ii) $D\{E[V_\propto(P:Q)]\} = 0 \quad \Longrightarrow P = Q$ (4.1.6)

(iii) $D[V_\propto(P:Q)] \geq 0$

(vi) $D\{E[V_\propto(P:Q)]\} \geq 0$ (4.1.7)

(v) $D[V_\propto(P:Q)]$ and $D\{E[V_\propto(P:Q)]\}$ are convex functions of $p_1, p_2, \dots \dots p_n$

Where Q is an a priori probability distribution which meets the constraints $V_\propto$(P: Q) is a convex function of p_1,p_2-----,p_n and q_1,q_2------,q_n and attains its maximum value zero when P = Q .

4.2 Measures of Symmetric Directed Divergence

Since, $D(P:Q) \neq D(Q:P)$

The measure $J(P:Q)$ is a symmetric divergence of P from Q defined by Kapur [1]

$J(P:Q) = D(P:Q) + D(Q:P)$ (4.2.1)

Since, $D(P:Q) \neq D(Q:P)$

So, $J(P:Q) = J(Q:P)$

Similarly, the symmetric directed divergence measure $J(P:Q)$ corresponding to $V_\alpha(P)$

$$D[V_\propto(P:Q)] = \frac{\sum_{i=1}^{n}(p_i-a_i)^\propto(q_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-p_i)^\propto(b_i-q_i)^{1-\propto}-\sum_{i=1}^{n}(b_i-a_i)}{\propto-1}$$

$$D[V_\propto(Q:P)] = \frac{\sum_{i=1}^{n}(q_i-a_i)^\propto(p_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-q_i)^\propto(b_i-p_i)^{1-\propto}-\sum_{i=1}^{n}(b_i-a_i)}{\propto-1}$$

$J[V_\propto(P:Q)] = D[V_\propto(P:Q)] + D[V_\propto(Q:P)]$

$$J[V_\propto(P:Q)] = \frac{1}{(\alpha-1)}[\sum_{i=1}^{n}(p_i-a_i)^\propto(q_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-p_i)^\propto(b_i-q_i)^{1-\propto}-2\sum_{i=1}^{n}(b_i-a_i)+\sum_{i=1}^{n}(q_i-a_i)^\propto(p_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-q_i)^\propto(b_i-p_i)^{1-\propto}]$$

where $\alpha > 0$ (4.2.2)

The symmetric directed divergence measure $J(P:Q)$ corresponding to $E[V_\alpha(P)]$

$J\{E[V_\propto(P:Q)]\} = E[V_\propto(P:Q)] + E[V_\propto(Q:P)]$

$$J\{E[V_\propto(P:Q)]\} = \frac{1}{(1-\alpha)}[\sum_{i=1}^{n}(p_i-a_i)^\propto(q_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-p_i)^\propto(b_i-q_i)^{1-\propto}-\sum(b_i-a_i)+\sum_{i=1}^{n}(q_i-a_i)^\propto(p_i-a_i)^{1-\propto}+\sum_{i=1}^{n}(b_i-q_i)^\propto(b_i-p_i)^{1-\propto}-\sum(b_i-a_i)]$$

where $\alpha < 0$ (4.2.3)

$J[V_\propto(P:Q)]$ and $J\{E[V_\propto(P:Q)]\}$ are the symmetric directed divergence measures of $V_\propto$(P: Q)and $E[V_\propto(P:Q)]$ measures of entropy.

4.3. Measures of Inaccuracy

The concept of inaccuracy has been initiated through coding theory thereby we can create measure of inaccuracy similar to every measure of entropy. We obtain a mean code word length similar to that mean code word length enabling us to have a measure of inaccuracy.

Kerridge [8] for the first time has defined measure of inaccuracy $I(P:Q)$ a measure of complete variability including uncertainty due to Q being different from P , he has enlisted the three essential properties of a measure of inaccuracy these are :

(i) $I(P:P)$ Should be a measure of entropy

(ii) $I(P:Q) \geq I(P:P)$, the equality sign holds only when $Q = P$

(iii) $I(P:Q) - I(P:P)$ Should represent some measure of directed divergence of P from Q

Examining entropy through coding theory approach, Kapur [9] has recommend the inaccuracy of the entropy H(P) of the distribution P relative to Q is given by

I(P:Q) = H(P) + D(P:Q) (4.3.1)

Where $H(P)$ is a measure of entropy, $D(P:Q)$ is a measures of directed divergence and $I(P:Q)$ is the measure of inaccuracy corresponding to $H(P)$.

From (4.3.1) we can write

$I[V_\propto(\mathrm{P{:}\,Q})] = [V_\propto(\mathrm{P})] + D[V_\propto(\mathrm{P{:}\,Q})]$

Similarly, where $[V_\propto(\mathrm{P})]$ is a measure of entropy, $D[V_\propto(P{:}Q)]$ is a measures of directed divergence and $I[V_\propto(P{:}\,Q)]$ is the measure of inaccuracy similar to measure of entropy $[V_\propto(\mathrm{P})]$.

It is obvious that $I[V_\propto(P{:}\,Q)]$ meet all the three properties itemized by Kerridge [4] so in the present case measure of Inaccuracy $I[V_\propto(\mathrm{P{:}\,Q})]$ is

$I[V_\propto(\mathrm{P{:}\,Q})] = [V_\propto(\mathrm{P})] + D[V_\propto(\mathrm{P{:}\,Q})]$

$$=\frac{\sum_{i=1}^{n}(p_i-a_i)^{\propto}\{1-(q_i-a_i)^{1-\alpha}\}+\sum_{i=1}^{n}(b_i-p_i)^{\propto}\{1-(b_i-q_i)^{1-\alpha}\}}{(1-\alpha)}, \text{ where } \alpha > 0 \quad (4.3.2)$$

Again from (4.3.1), we can write

$I\{E[V_\propto(\mathrm{P{:}\,Q})]\} = \{E[V_\propto(\mathrm{P})]\} + D\{E[V_\propto(\mathrm{P{:}\,Q})]\}$

Where $E[V_\propto(\mathrm{P})]$ is a measure of entropy, $D\{E[V_\propto(P{:}\,Q)]\}$ is a measures of directed divergence and $I[V_\propto(P{:}\,Q)]$ is the measures of inaccuracy corresponding to measure of entropy $E[V_\propto(\mathrm{P})]$.

It is apparent that $I\{E[V_\propto(P{:}Q)]\}$ meets all the three properties defined by Kerridge [8], hence in the present case measure of Inaccuracy $I[V_\propto(\mathrm{P{:}\,Q})]$ is

$I\{E[V_\propto(\mathrm{P{:}\,Q})]\} = E[V_\propto(\mathrm{P})] + D\{E[V_\propto(\mathrm{P{:}\,Q})]\}$

$$=\frac{\sum_{i=1}^{n}(p_i-a_i)^{\propto}\{1-(q_i-a_i)^{1-\alpha}\}+\sum_{i=1}^{n}(b_i-p_i)^{\propto}\{1-(b_i-q_i)^{1-\alpha}\}}{(\alpha-1)},$$

where α < 0 (4.3.3)

4.4 Measures of Information Improvement

Let P = $(p_1, p_2, \dots, p_n)$ be the true or the ideal distribution. Let Q= $(q_1, q_2, \dots, q_n)$ be the original estimate or realization of P and let R=$(r_1, r_2, \dots, r_m)$ be the modified approximation or apprehension of P. The directed divergences of P from Q are D(P:Q) and the directed divergences of P from R are D(P:R) respectively. If D(P:R) is smaller than D(P:Q) then the revised distribution is close to the true distribution. Subsequently, in this regard, Theil [10] has suggested measures of information improvement

I(P:Q:R) = D(P:Q) – D(P:R) (4.4.1)

A measures of information improvement I(P:Q:R) may be positive or negative depending on whether the revised estimate is closer or remote from P than the original estimate.

Obviously, it is understood that I (P,Q,R) is permutationally symmetric i.e. does not change when n triplets (p_i, q_i, r_i) where i = 1,2,…..n are permuted among themselves.

Where P is true distribution, Q is observed distribution and R is reassessed distribution with following properties,

$p_i \geq 0\ q_i \geq 0$ and $r_i \geq 0$ (4.4.2)

$a_i \leq p_i \leq b_i, a_i \leq q_i \leq b_i, a_i \leq r_i \leq b_i$ (4.4.3)

$\sum p_i = \sum q_i = \sum r_i = 1$

$I[V_\propto(\mathrm{P{:}\,Q{:}R})] = \mathrm{D}[V_\propto(\mathrm{P{:}\,Q})] - D[V_\propto(\mathrm{P{:}\,R})]$

$I[V_\propto(\mathrm{P{:}\,Q{:}R})] = \frac{1}{(\alpha-1)}\sum_{i=1}^{n}(p_i-a_i)^{\propto}[(q_i-a_i)^{1-\propto}-(r_i-a_i)^{1-\propto}]+\sum_{i=1}^{n}(b_i-p_i)^{\propto}[(b_i-q_i)^{1-\propto}-(b_i-r_i)^{1-\propto}]$

$\mathrm{I}[V_\propto(\mathrm{S{:}\,Q{:}\,R})] = \frac{1}{(\alpha-1)}\sum_{i=1}^{n}(s_i-a_i)^{\propto}[(q_i-a_i)^{1-\propto}-(r_i-a_i)^{1-\propto}]+\sum_{i=1}^{n}(b_i-s_i)^{\propto}[(b_i-q_i)^{1-\propto}-(b_i-r_i)^{1-\propto}]$, where $\alpha > 0$

(4.4.4)

$I[V_\propto(\text{P: Q:R})], \text{I}[V_\propto(\text{S: Q: R})]$ are a measures of information improvement corresponding to $V_\propto(\text{P})$ measure of entropy.

Similarly,

$$I\{E[V_\propto(\text{P: Q:R})] = \frac{1}{(1-\alpha)}\sum_{i=1}^{n}(p_i - a_i)^\propto[(q_i - a_i)^{1-\propto} - (r_i - a_i)^{1-\propto}] + \sum_{i=1}^{n}(b_i - p_i)^\propto[(b_i - q_i)^{1-\propto} - (b_i - r_i)^{1-\propto}]$$

$$I\{E[V_\propto(\text{S: Q: R})]\} = \frac{1}{(1-\alpha)}\sum_{i=1}^{n}(s_i - a_i)^\propto[(q_i - a_i)^{1-\propto} - (r_i - a_i)^{1-\propto}] + \sum_{i=1}^{n}(b_i - s_i)^\propto[(b_i - q_i)^{1-\propto} - (b_i - r_i)^{1-\propto}]$$

where $\alpha < 0$ (4.4.5)

$I\{E[V_\propto(\text{P: Q:R})]\}, I\{E[V_\propto(\text{S: Q:R})]\}$ are measures of information improvement corresponding to $E[V_\propto(\text{P})]$ measure of entropy.

4.5 Measure of Generlised Information Improvement:

At times, during the investigation true distribution P is also changed to another distribution S then measures of generlised information enhancement is given by

$$G(P: S/Q/R) = \frac{I(P:Q:R)+I(S:Q:R)}{2} \quad (4.5.1)$$

Where I(P:Q:R) = D(P:Q) – D(P:R) (4.5.2)

And I (S: Q: R) = D(S: Q) – D(S: R) (4.5.3)

Where there is no change in true distribution i.e. S=P

G(P:P/Q/R) = I(P:Q:R)

In the present case

$$G[V\alpha(P:S/Q/R)] = \frac{I[V_\propto(P:Q:R)]+I[V_\propto(S:Q:R)]}{2}$$

$I[V_\propto(\text{P: Q:R})] = \text{D}\,[V_\propto(\text{P: Q})] - D\,[V_\propto(\text{P: R})]$

$I[V_\propto(\text{S: Q: R})] = \text{D}[V_\propto(\text{S: Q})] - D\,[V_\propto(\text{S: R})]$

$$G[V\alpha(P:S/Q/R)] = \frac{\text{D}[V_\propto(\text{P: Q})] - D[V_\propto(\text{P: R})] + \text{D}[V_\propto(\text{S: Q})] - D[V_\propto(\text{S: R})]}{2}$$

$$G[V\alpha(P:S/Q/R)] = \frac{1}{\alpha-1}\left[\sum\left\{\frac{(p_i-a_i)^\alpha+(s_i-a_i)^\alpha}{2}\right\}\{(q_i - a_i)^{1-\alpha} + (r_i - a_i)^{1-\alpha}\} + \sum\left\{\frac{(b_i-p_i)^\alpha+(b_i-s_i)^\alpha}{2}\right\}\{(b_i - q_i)^{1-\alpha} + (b_i - r_i)^{1-\alpha}\}\right]$$

where $\alpha > 0$ (4.5.4)

$G[V\alpha(P:S/Q/R)]$ is a measure of generlised information improvement corresponding to $V_\propto(\text{P})$ measure of entropy.

Again we take

$$G\{E[V\alpha(P:S/Q/R)]\} = \frac{I\{E[V_\propto(P:Q:R)]\} + I\{E[V_\propto(S:Q:R)]\}}{2}$$

$$G\{E[V\alpha(P:S/Q/R)]\} = \frac{1}{1-\alpha}\left[\sum\left\{\frac{(p_i-a_i)^\alpha+(s_i-a_i)^\alpha}{2}\right\}\{(q_i - a_i)^{1-\alpha} + (r_i - a_i)^{1-\alpha}\} + \sum\left\{\frac{(b_i-p_i)^\alpha+(b_i-s_i)^\alpha}{2}\right\}\{(b_i - q_i)^{1-\alpha} + (b_i - r_i)^{1-\alpha}\}\right]$$

$\alpha < 0$ (4.5.5)

$G\{E[V\alpha(P:S/Q/R)]\}$ is a measure of generalized information improvement corresponding to $E[V_\propto(\text{P})]$ measure of entropy.

V. CONCLUSION

- In this paper, we have recommended measures of entropy specifying the infirmities of Kapur [1] measure of entropy, and examining the various properties of proposed measure of entropy. The proposed measures are reassessed from Shannon [2] measure of entropy. Our modified entropy has better range of validity, when there are inequality constraints of type $a_i \leq p_i \leq b_i$, a ≥ 0, b ≤ 1 on probabilities. And it is verified that there are no addition constraints except the natural constraints $\sum_{i=1}^{n} p_i = 1$ then the maximum value of measure of entropy is concave function.

- The proposed measures of entropy meets all the properties of measures of entropy. The maximum value of measures of entropy is conditional to natural constraints. It is also authentically concave function of $p_1, p_2, \ldots\ldots p_n$, so, when $V_\propto(P)$ and $E[V_\propto(P)]$ are maximized conditional to linear constraints (1) and (2) their local maximum will be their global maximum. Therefore, we can mention that we have obtained the globally measures of entropy
- We derived the measures of directed divergence, measures of symmetric directed divergence, measures of inaccuracy, measures of information improvement and measures of generalized information improvement corresponding to the new introduced measures of entropy
- The main advantageous primacy of proposed measures of entropy maximization problem over other proposed measures of entropy is very less owing to these axioms which are used to derive it.
- The significant application of the new measures of entropy and directed divergence is that it has relevance in the city population distribution model, for cost change. Likewise, in the transportation model, a power station is required to supply electric power between certain minimum and maximum number of units to various consumers with various limits. For example, if a person has 'n' number of representative to be assigned to 'm' number of stores. Each store owner requires a minimum and maximum number of representative. The incomes per representative are $p_1, p_2, \ldots p_n$ from the different stores, in this case the proposed measures may be useful.
- With its various application in various fields, the proposed measures may be relevantly functional in other distribution problems. If there exists the upper and lower bounds for each individual allocation, these measures may be applicable such as in allotment of students to schools, patient and doctors to hospitals, resources to different objects, electricity to different groups users etc.

  We have the objective of its extensive applications; however, we also fancy for optimum target to be met with in each individual case.

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