IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 08, Issue 7 (July. 2018), ||V (VI) || 28-37

On the Bounds for the Zeros of a Polynomial

M.H.Gulzar[1], Ajaz Wani[2], Rubia Akhter[3]
Department of Mathematics, University of Kashmir, Srinagar (India) 190006.
Corresponding author: M.H.Gulzar[1],

Abstract. In this paper we find bounds for the zeros of a complex polynomial when the coefficients of the polynomial are restricted to certain conditions.
Mathematics Subject Classification (2010): 30C10, 30C15.

Key Words and Phrases: Coefficient, Open Disc, Polynomial, Zeros.

Date of Submission: 12-07-2018 Date of acceptance: 28-07-2018

I. INTRODUCTION

In connection with the bounds for the zeros of a polynomial with real coefficients, Gulzar [3] recently proved the following results.

Theorem A. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0<\rho \leq 1, 0<\lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 .$$

Then all the zeros of P(z) lie in the closed disk

$$|z+k_1-1| \leq \frac{1}{|a_n|}\{k_1 a_n + 2(k_2-1)|a_\lambda| - \rho(a_0+|a_0|) + 2|a_0|\}.$$

Theorem B. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0<\rho \leq 1, 0<\lambda \leq n-1$,

$$k_1 a_n \geq a_{n-1} \geq ...... \geq k_2 a_\lambda \geq a_{\lambda-1} \geq ...... \geq a_1 \geq \rho a_0 .$$

Then for any R>0,the number of zeros of P(z) in $\frac{|a_0|}{M} \leq |z| \leq \frac{R}{c}, c>1$ does not exceed

$$\frac{1}{\log c}\log\frac{K}{|P(0)|},$$

where

$$K=|a_n|R^{n+1}+|a_0|+R^n[\ k_1(|a_n|+a_n)-|a_n|+2(k_2-1)|a_\lambda|+|a_0|-\rho(|a_0|+a_0)\ ],$$

$$M=|a_n|R^{n+1}+R^n[\ k_1(|a_n|+a_n)-|a_n|+2(k_2-1)|a_\lambda|+|a_0|-\rho(|a_0|+a_0)\ ]$$

for $R \geq 1$ and

$$K=|a_n|R^{n+1}+|a_0|+R[\ k_1(|a_n|+a_n)-|a_n|+2(k_2-1)|a_\lambda|+|a_0|-\rho(|a_0|+a_0)\ ],$$

$$M=|a_n|R^{n+1}+R[\ k_1(|a_n|+a_n)-|a_n|+2(k_2-1)|a_\lambda|+|a_0|-\rho(|a_0|+a_0)\ ]$$

for $R \leq 1$.

II. MAIN RESULTS

The aim of this paper is to consider the polynomial of Theorem A with complex coefficients and find bounds for its zeros and for the number of its zeros in a specific region. In fact, we prove the following results.

Theorem 1. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1, k_2, ρ and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0<\rho \leq 1, 0<\lambda \leq n-1$,

$$k_1|a_n| \geq |a_{n-1}| \geq \ldots\ldots \geq k_2|a_\lambda| \geq |a_{\lambda-1}| \geq \ldots\ldots \geq |a_1| \geq \rho|a_0|$$

and for some real numbers α, β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j=0,1,2,\ldots\ldots,n.$$

Then P(z) has all its zeros in the closed disk

$$|z+k_1-1| \leq \frac{1}{|a_n|}\Big\{\ k_1|a_n|(\cos\alpha+\sin\alpha)+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$+2|a_0|-\rho|a_0|(\cos\alpha-\sin\alpha+1)+2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1}|a_j|\ \Big\}.$$

Remark 1. Choosing $\alpha=0, \beta=0$ and $a_j \geq 0, \forall j$ in Theorem 1, we get Theorem A.

For different values of the parameters , we get many different results. For example taking $k_2=1$ in Theorem 1, we get the following result.

Corollary 1. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1, ρ and some integer λ with $k_1 \geq 1, 0<\rho \leq 1, 0<\lambda \leq n-1$,

$$k_1|a_n| \geq |a_{n-1}| \geq \ldots\ldots \geq |a_\lambda| \geq |a_{\lambda-1}| \geq \ldots\ldots \geq |a_1| \geq \rho|a_0|$$

and for some real numbers α, β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j=0,1,2,\ldots\ldots,n.$$

Then P(z) has all its zeros in the closed disk

$$|z+k_1-1| \leq \frac{1}{|a_n|}\Big\{\ k_1|a_n|(\cos\alpha+\sin\alpha)+2|a_0|-\rho|a_0|(\cos\alpha-\sin\alpha+1)$$
$$+2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1}|a_j|\ \Big\}.$$

Taking $\rho=1$ in Theorem 1, we get the following result.

Corollary 2. Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1, k_2 and some integer λ with $k_1 \geq 1, k_2 \geq 1, 0<\lambda \leq n-1$,

$$k_1|a_n| \geq |a_{n-1}| \geq \ldots\ldots \geq k_2|a_\lambda| \geq |a_{\lambda-1}| \geq \ldots\ldots \geq |a_1| \geq |a_0|$$

and for some real numbers α, β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j=0,1,2,\ldots\ldots,n.$$

Then P(z) has all its zeros in the closed disk

$$|z+k_1-1| \leq \frac{1}{|a_n|}\Big\{\ k_1|a_n|(\cos\alpha+\sin\alpha)+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$

$$-|a_0|(\cos\alpha-\sin\alpha-1)+2\sin\alpha\sum_{j=1,j\neq\lambda}^{n-1}|a_j|\ \}.$$

Next, we prove the following result.

Theorem 2. Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1,k_2,ρ and some integer λ with $k_1\geq 1, k_2\geq 1, 0<\rho\leq 1, 0<\lambda\leq n-1$,

$$k_1|a_n|\geq|a_{n-1}|\geq......\geq k_2|a_\lambda|\geq|a_{\lambda-1}|\geq......\geq|a_1|\geq\rho|a_0|$$

and for some real numbers α,β,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,2,......,n.$$

Then P(z) has no zeros in the open disk $|z|<\frac{|a_0|}{M}$,

where

$$M=|a_n|R^{n+1}+R^n[\ (k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\neq\lambda}^{n-1}|a_j|\Bigg]$$
$$\text{for } R\geq 1$$
$$=|a_n|R^{n+1}+R[\ (k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\neq\lambda}^{n-1}|a_j|\Bigg]$$
$$\text{for } R\leq 1.$$

For different values of the parameters, we get many different results. For example taking $k_2=1$ in Theorem 2, we get the following result.

Corollary 3. Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1,ρ and some integer λ with $k_1\geq 1, 0<\rho\leq 1, 0<\lambda\leq n-1$,

$$k_1|a_n|\geq|a_{n-1}|\geq......\geq|a_\lambda|\geq|a_{\lambda-1}|\geq......\geq|a_1|\geq\rho|a_0|$$

and for some real numbers α,β,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2},\ j=0,1,2,......,n.$$

Then P(z) has no zeros in the open disk $|z|<\frac{|a_0|}{M}$,

where

$$M=|a_n|R^{n+1}+R^n[\ (k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|$$
$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,}^{n-1}|a_j|\ ]$$
$$\text{for } R\geq 1$$
$$=|a_n|R^{n+1}+R[\ (k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|$$

$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,}^{n-1}|a_j| \;]$$

for $R\leq 1$.

Taking $\rho=1$ in Theorem 2, we get the following result.

Corollary 4. Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1,k_2 and some integer λ with $k_1\geq 1, k_2\geq 1, 0<\lambda\leq n-1$,

$$k_1|a_n|\geq|a_{n-1}|\geq......\geq k_2|a_\lambda|\geq|a_{\lambda-1}|\geq......\geq|a_1|\geq|a_0|$$

and for some real numbers α,β,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2}, j=0,1,2,......,n.$$

Then P(z) has no zeros in the open disk $|z|<\frac{|a_0|}{M}$,

where

$$M=|a_n|R^{n+1}+R^n\Big[\;(k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-|a_0|(\cos\alpha-\sin\alpha)+2\sin\alpha\sum_{j=1,j\neq\lambda}^{n-1}|a_j|\Big]$$

for $R\geq 1$

$$=|a_n|R^{n+1}+R\Big[\;(k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-|a_0|(\cos\alpha-\sin\alpha)+2\sin\alpha\sum_{j=1,j\neq\lambda}^{n-1}|a_j|\Big]$$

for $R\leq 1$.

Next we prove

Theorem 3. Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1,k_2,ρ and some integer λ with $k_1\geq 1, k_2\geq 1, 0<\rho\leq 1, 0<\lambda\leq n-1$,

$$k_1|a_n|\geq|a_{n-1}|\geq......\geq k_2|a_\lambda|\geq|a_{\lambda-1}|\geq......\geq|a_1|\geq\rho|a_0|$$

and for some real numbers α,β,

$$|\arg a_j-\beta|\leq\alpha\leq\frac{\pi}{2}, j=0,1,2,......,n.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M}\leq|z|\leq\frac{R}{c}, c>1$ is less than or equal to

$$\frac{1}{\log c}\log\frac{K}{|a_0|},$$

where

$$M=|a_n|R^{n+1}+R^n\Big[\;(k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\neq\lambda}^{n-1}|a_j|\Big]$$

for $R\geq 1$

$$= |a_n| R^{n+1} + R\Big[\, (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2 - 1)|a_\lambda| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\Big] \quad \text{for } R \le 1,$$

$$K = |a_n| R^{n+1} + R^n\Big[\, (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2 - 1)|a_\lambda| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\Big] \quad \text{for } R \ge 1$$

$$= |a_n| R^{n+1} + R\Big[\, (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2 - 1)|a_\lambda| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\Big] \quad \text{for } R \le 1.$$

For different values of the parameters, we get many different results. For example taking $\rho = 1$ in Theorem 2, we get the following result.

Corollary 5. Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some positive numbers k_1, k_2 and some integer λ with $k_1 \ge 1, k_2 \ge 1, 0 < \lambda \le n-1$,

$$k_1|a_n| \ge |a_{n-1}| \ge \ldots\ldots \ge k_2|a_\lambda| \ge |a_{\lambda-1}| \ge \ldots\ldots \ge |a_1| \ge |a_0|$$

and for some real numbers α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,2,\ldots\ldots,n.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{M} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to

$$\frac{1}{\log c} \log \frac{K}{|a_0|},$$

where

$$M = |a_n| R^{n+1} + R^n\Big[\, (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2 - 1)|a_\lambda| - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\Big] \quad \text{for } R \ge 1$$

$$= |a_n| R^{n+1} + R\Big[\, (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2 - 1)|a_\lambda| - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\Big] \quad \text{for } R \le 1,$$

$$K = |a_n| R^{n+1} + |a_0| + R^n\Big[\, (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2 - 1)|a_\lambda| - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\Big]$$

for $R \geq 1$

$$= |a_n| R^{n+1} + |a_0| + R\Big[\ (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2-1)|a_\lambda| - |a_0|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\Big]$$

for $R \leq 1$.

III. LEMMAS

For the proofs of the above results, we need the following lemmas.

Lemma 1. Let f(z) (not identically zero) be analytic for $|z| \leq R,\ f(0) \neq 0$ and $f(a_k) = 0$, $k = 1,2,......,n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{Re}^{i\theta}|d\theta - \log|f(0)| = \sum_{j=1}^{n} \log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 2. Let f (z) be analytic for $|z| \leq R,\ f(0) \neq 0$ and $|f(z)| \leq M$ for $|z| \leq R$. Then the number of zeros of f(z) in $|z| \leq \frac{R}{c}, c > 1$ does not exceed $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 2.

Lemma3. For any two complex numbers z_1, z_2 such that $|z_1| \geq |z_2|$ and for some real α, β, $|\arg z_j - \beta| \leq \alpha \leq \frac{\pi}{2}, j = 1,2$, we have

$$|z_1 - z_2| \leq (|z_1| - |z_2|)\cos\alpha + (|z_1| + |z_2|)\sin\alpha.$$

Lemma 3 is due to Govil and Rahman [2].

IV, PROOFS OF THEOREMS

Proof of Theorem 1. Consider the polynomial

$$F(z) = (1-z)P(z)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + ... + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + (k_1 a_n - a_{n-1})z^n - (k_1 - 1)a_n z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{\lambda+1} - k_2 a_\lambda)z^{\lambda+1}$$
$$+ (k_2 - 1)a_\lambda z^{\lambda+1} + (k_2 a_\lambda - a_{\lambda-1})z^\lambda - (k_2 - 1)a_\lambda z^\lambda + (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1} + ...$$
$$+ (a_2 - a_1)z^2 + (a_1 - \rho a_0)z + (\rho - 1)a_0 z\} + a_0$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, j = 1,2,....,n$, we have by using the hypothesis and Lemma 3,

$$|F(z)| \geq |a_n||z^n|\Big[\ |z + k_1 - 1| - \frac{1}{|a_n|}\Big\{\ |k_1 a_n - a_{n-1}| + |a_{n-1} - a_{n-2}|.\frac{1}{|z|} + ... + |a_{\lambda+1} - k_2\alpha_\lambda|.\frac{1}{|z|^{n-\lambda-1}}$$
$$+ (k_2 - 1)|a_\lambda|.\frac{1}{|z|^{n-\lambda-1}} + |k_2 a_\lambda - a_{\lambda-1}|.\frac{1}{|z|^{n-\lambda}} + (k_2 - 1)|a_\lambda|.\frac{1}{|z|^{n-\lambda}}$$
$$+ |a_{\lambda-1} - a_{\lambda-2}|.\frac{1}{|z|^{n-\lambda+1}} + ... + |a_2 - a_1|.\frac{1}{|z|^{n-1}} + |a_1 - \rho a_0|.\frac{1}{|z|^{n-1}} + (1-\rho)|a_0|.\frac{1}{|z|^{n-1}}$$
$$+ |a_0|.\frac{1}{|z|^n}\ \Big\}\ \Big]$$

$$> |a_n||z^n|\Big[\ |z+k_1-1| - \frac{1}{|a_n|}\Big\{\ |k_1 a_n - a_{n-1}| + |a_{n-1} - a_{n-2}|. + ... + |a_{\lambda+1} - k_2\alpha_\lambda|.$$

$$+ (k_2-1)|a_\lambda|. + |k_2 a_\lambda - a_{\lambda-1}|. + (k_2-1)|a_\lambda|. + |a_{\lambda-1} - a_{\lambda-2}|. + ...$$

$$+ |a_2 - a_1|. + |a_1 - \rho a_0|. + (1-\rho)|a_0| + |a_0|\ \Big\}\ \Big]$$

$$\geq |a_n||z^n|\Big[\ |z+k_1-1| - \frac{1}{|a_n|}\Big\{\ (k_1|a_n| - |a_{n-1}|)\cos\alpha + (k_1|a_n| + |a_{n-1}|)\sin\alpha$$

$$+ (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + ...$$

$$+ (|a_{\lambda+1}| - k_2|a_\lambda|)\cos\alpha + (|a_{\lambda+1}| + k_2|a_\lambda|)\sin\alpha + 2(k_2-1)|a_\lambda|$$

$$+ (k_2|a_\lambda| - |a_{\lambda-1}|)\cos\alpha + (k_2|a_\lambda| + |a_{\lambda-1}|)\sin\alpha + (|a_{\lambda-1}| - |a_{\lambda-2}|)\cos\alpha$$

$$+ (|a_{\lambda-1}| + |a_{\lambda-2}|)\sin\alpha + ... + (|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha$$

$$+ (|a_1| - \rho|a_0|)\cos\alpha + (|a_1| + \rho|a_0|)\sin\alpha + (1-\rho)|a_0| + |a_0|\ \Big\}\ \Big]$$

$$\geq |a_n||z|^n\Big[\ |z+k_1-1| - \frac{1}{|a_n|}\Big\{\ k_1|a_n|(\cos\alpha + \sin\alpha) + 2k_2|a_\lambda|\sin\alpha + 2(k_2-1)|a_\lambda|$$

$$+ 2|a_0| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1}|a_j|\ \Big\}\ \Big]$$

$$> 0$$

if

$$|z+k_1-1| > \frac{1}{|a_n|}\Big\{\ k_1|a_n|(\cos\alpha + \sin\alpha) + 2k_2|a_\lambda|\sin\alpha + 2(k_2-1)|a_\lambda|$$

$$+ 2|a_0| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1}|a_j|\ \Big\}.$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in the closed disk

$$|z+k_1-1| \leq \frac{1}{|a_n|}\Big\{\ k_1|a_n|(\cos\alpha + \sin\alpha) + 2k_2|a_\lambda|\sin\alpha + 2(k_2-1)|a_\lambda|$$

$$+ 2|a_0| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1}|a_j|\ \Big\}.$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality, it follows that all the zeros of F(z) and hence P(z) lie in the closed disk

$$|z+k_1-1| \leq \frac{1}{|a_n|}\Big\{\ k_1|a_n|(\cos\alpha + \sin\alpha) + 2k_2|a_\lambda|\sin\alpha + 2(k_2-1)|a_\lambda|$$

$$+ 2|a_0| - \rho|a_0|(\cos\alpha - \sin\alpha + 1) + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1}|a_j|\ \Big\},$$

thereby proving Theorem 1.

Proof of Theorem2. For the polynomial

$$F(z) = (1-z)P(z)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + ... + (a_1 - a_0)z + a_0$$

$$= -a_n z^{n+1} + (k_1 a_n - a_{n-1})z^n - (k_1-1)a_n z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_{\lambda+1} - k_2 a_\lambda)z^{\lambda+1}$$

$$+(k_2-1)a_\lambda z^{\lambda+1}+(k_2a_\lambda-a_{\lambda-1})z^\lambda-(k_2-1)a_\lambda z^\lambda+(a_{\lambda-1}-a_{\lambda-2})z^{\lambda-1}+.....$$
$$+(a_2-a_1)z^2+(a_1-\rho a_0)z+(\rho-1)a_0z.+a_0$$
$$=a_0+G(z)$$

where

$$G(z)=-a_nz^{n+1}+(k_1a_n-a_{n-1})z^n-(k_1-1)a_nz^n+(a_{n-1}-a_{n-2})z^{n-1}+....+(a_{\lambda+1}-k_2a_\lambda)z^{\lambda+1}$$
$$+(k_2-1)a_\lambda z^{\lambda+1}+(k_2a_\lambda-a_{\lambda-1})z^\lambda-(k_2-1)a_\lambda z^\lambda+(a_{\lambda-1}-a_{\lambda-2})z^{\lambda-1}+.....$$
$$+(a_2-a_1)z^2+(a_1-\rho a_0)z+(\rho-1)a_0z.$$

for $|z|\le R$, we have by using the hypothesis and Lemma 3,

$$|G(z)|\le|a_n|R^{n+1}+|(k_1-1)a_n|R^n+|k_1a_n-a_{n-1}|R^n+|a_{n-1}-a_{n-2}|.R^{n-1}+....+|a_{\lambda+1}-k_2a_\lambda|.R^{\lambda+1}$$
$$+|(k_2-1)a_\lambda|.R^{\lambda+1}+|k_2a_\lambda-a_{\lambda-1}|.R^\lambda+|(k_2-1)a_\lambda|.R^\lambda$$
$$+|a_{\lambda-1}-a_{\lambda-2}|.R^{\lambda-1}+.....+|a_2-a_1|.R^2+|a_1-\rho a_0|.R+|(\rho-1)a_0|.R$$
$$\le|a_n|R^{n+1}+R^n[\ (k_1-1)|a_n|+|k_1a_n-a_{n-1}|+|a_{n-1}-a_{n-2}|+......+|a_{\lambda+1}-k_2a_\lambda|$$
$$+(k_2-1)|a_\lambda|+|k_2a_\lambda-a_{\lambda-1}|+(k_2-1)|a_\lambda|+|a_{\lambda-1}-a_{\lambda-2}|+.....$$
$$+|a_2-a_1|+|a_1-\rho a_0|+(1-\rho)|a_0|\ ]$$
$$\le|a_n|R^{n+1}+R^n[\ (k_1-1)|a_n|+(k_1|a_n|-|a_{n-1}|)\cos\alpha+(|k_1a_n|+|a_{n-1}|)\sin\alpha$$
$$+(|a_{n-1}|-|a_{n-2}|)\cos\alpha+(|a_{n-1}|+|a_{n-2}|)\sin\alpha+......$$
$$+(|a_{\lambda+1}|-k_2|a_\lambda|)\cos\alpha+(|a_{\lambda+1}|+k_2|a_\lambda|)\sin\alpha$$
$$+2(k_2-1)|a_\lambda|+(k_2|a_\lambda|-|a_{\lambda-1}|)\cos\alpha+(k_2|a_\lambda|+|a_{\lambda-1}|)\sin\alpha$$
$$+(|a_{\lambda-1}|-|a_{\lambda-2}|)\cos\alpha+(|a_{\lambda-1}|+|a_{\lambda-2}|)\sin\alpha+.....$$
$$+(|a_2|-|a_1|)\cos\alpha+(|a_2|+|a_1|)\sin\alpha+(|a_1|-\rho|a_0|)\cos\alpha$$
$$+(|a_1|+\rho|a_0|)\sin\alpha+(1-\rho)|a_0|\ ]$$
$$=|a_n|R^{n+1}+R^n[\ (k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\ne\lambda}^{n-1}|a_j|\ ]$$

for $R\ge1$.

For $R\le1$,

$$|G(z)|\le|a_n|R^{n+1}+R[\ (k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\ne\lambda}^{n-1}|a_j|\ ].$$

Since G(z) is analytic for $|z|\le R$ and G(0)=0, it follows by Schwarz lemma that

$|G(z)|\le M|z|$ in $|z|\le R$, where

$$M=|a_n|R^{n+1}+R^n[\ (k_1|a_n|(\cos\alpha+\sin\alpha+1)-|a_n|+2k_2|a_\lambda|\sin\alpha+2(k_2-1)|a_\lambda|$$
$$-\rho|a_0|(\cos\alpha-\sin\alpha+1)+|a_0|+2\sin\alpha\sum_{j=1,j\ne\lambda}^{n-1}|a_j|\ ]$$

for $R\ge1$.

$$= |a_n| R^{n+1} + R^n [\ (k_1|a_n|(\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2|a_\lambda|\sin\alpha + 2(k_2 - 1)|a_\lambda|$$
$$- \rho|a_0|(\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j|\ ]$$
for $R \leq 1$.

Hence ,

$$|F(z)| = |a_0 + G(z)|$$
$$\geq |a_0| - |G(z)|$$
$$\geq |a_0| - M|z|$$
$$> 0$$

if $|z| < \frac{|a_0|}{M}$.

In other words, no zero of F(z) lies in $|z| < \frac{|a_0|}{M}$.

Since the zeros of P(z) are also the zeros of F(z), it follows no zero of F(z) lies in $|z| < \frac{|a_0|}{M}$. That completes the proof of Theorem 2.

Proof of Theorem 3. For the polynomial

$$F(z) = (1-z)P(z)$$
$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \ldots\ldots + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda + \ldots\ldots + (a_1 - a_0)z + a_0$$
$$= -a_n z^{n+1} + (k_1 a_n - a_{n-1})z^n - (k_1 - 1)a_n z^n + (a_{n-1} - a_{n-2})z^{n-1} + \ldots. + (a_{\lambda+1} - k_2 a_\lambda)z^{\lambda+1}$$
$$+ (k_2 - 1)a_\lambda z^{\lambda+1} + (k_2 a_\lambda - a_{\lambda-1})z^\lambda - (k_2 - 1)a_\lambda z^\lambda + (a_{\lambda-1} - a_{\lambda-2})z^{\lambda-1} + \ldots..$$
$$+ (a_2 - a_1)z^2 + (a_1 - \rho a_0)z + (\rho - 1)a_0 z. + a_0,$$

for $|z| \leq R$, we have by using the hypothesis and Lemma 3,

$$|F(z)| \leq |a_n| R^{n+1} + |(k_1 - 1)a_n| R^n + |k_1 a_n - a_{n-1}| R^n + |a_{n-1} - a_{n-2}|.R^{n-1} + \ldots. + |a_{\lambda+1} - k_2 a_\lambda|.R^{\lambda+1}$$
$$+ |(k_2 - 1)a_\lambda|.R^{\lambda+1} + |k_2 a_\lambda - a_{\lambda-1}|.R^\lambda + |(k_2 - 1)a_\lambda|.R^\lambda$$
$$+ |a_{\lambda-1} - a_{\lambda-2}|.R^{\lambda-1} + \ldots.. + |a_2 - a_1|.R^2 + |a_1 - \rho a_0|.R + |(\rho - 1)a_0|.R + |a_0|$$
$$\leq |a_n| R^{n+1} + |a_0| + R^n [\ (k_1 - 1)|a_n| + |k_1 a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \ldots\ldots + |a_{\lambda+1} - k_2 a_\lambda|$$
$$+ (k_2 - 1)|a_\lambda| + |k_2 a_\lambda - a_{\lambda-1}| + (k_2 - 1)|a_\lambda| + |a_{\lambda-1} - a_{\lambda-2}| + \ldots..$$
$$+ |a_2 - a_1| + |a_1 - \rho a_0| + (1-\rho)|a_0|\ ]$$
$$\leq |a_n| R^{n+1} + |a_0| + R^n [\ (k_1 - 1)|a_n| + (k_1|a_n| - |a_{n-1}|)\cos\alpha + (|k_1 a_n| + |a_{n-1}|)\sin\alpha$$
$$+ (|a_{n-1}| - |a_{n-2}|)\cos\alpha + (|a_{n-1}| + |a_{n-2}|)\sin\alpha + \ldots\ldots$$
$$+ (|a_{\lambda+1}| - k_2|a_\lambda|)\cos\alpha + (|a_{\lambda+1}| + k_2|a_\lambda|)\sin\alpha$$
$$+ 2(k_2 - 1)|a_\lambda| + (k_2|a_\lambda| - |a_{\lambda-1}|)\cos\alpha + (k_2|a_\lambda| + |a_{\lambda-1}|)\sin\alpha$$
$$+ (|a_{\lambda-1}| - |a_{\lambda-2}|)\cos\alpha + (|a_{\lambda-1}| + |a_{\lambda-2}|)\sin\alpha + \ldots..$$
$$+ (|a_2| - |a_1|)\cos\alpha + (|a_2| + |a_1|)\sin\alpha + (|a_1| - \rho|a_0|)\cos\alpha$$
$$+ (|a_1| + \rho|a_0|)\sin\alpha + (1-\rho)|a_0|\ ]$$

$$= |a_n| R^{n+1} + |a_0| + R^n \Big[\ (k_1 |a_n| (\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2 |a_\lambda| \sin\alpha + 2(k_2 - 1)|a_\lambda|$$
$$- \rho |a_0| (\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j| \ \Big]$$

for $R \geq 1$.

For $R \leq 1$,

$$|F(z)| \leq |a_n| R^{n+1} + |a_0| + R \Big[\ (k_1 |a_n| (\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2 |a_\lambda| \sin\alpha + 2(k_2 - 1)|a_\lambda|$$
$$- \rho |a_0| (\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j| \ \Big].$$

Hence, by Lemma 2, the number of zeros of F(z) and hence P(z) in $|z| \leq \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c} \log \frac{K}{|a_0|}$,

where

$$K = |a_n| R^{n+1} + |a_0| + R^n \Big[\ (k_1 |a_n| (\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2 |a_\lambda| \sin\alpha + 2(k_2 - 1)|a_\lambda|$$
$$- \rho |a_0| (\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j| \Bigg] \quad \text{for } R \geq 1$$
$$= |a_n| R^{n+1} + |a_0| + R \Big[\ (k_1 |a_n| (\cos\alpha + \sin\alpha + 1) - |a_n| + 2k_2 |a_\lambda| \sin\alpha + 2(k_2 - 1)|a_\lambda|$$
$$- \rho |a_0| (\cos\alpha - \sin\alpha + 1) + |a_0| + 2\sin\alpha \sum_{j=1, j\neq\lambda}^{n-1} |a_j| \Bigg] \quad \text{for } R \leq 1.$$

By using Theorem 2, the result then follows.

REFERENCES

[1]. L. V. Ahlfors ,Complex Analysis, 3rd edition, Mc-Grawhill.,1979.

[2]. N.K.Govil and Q.I.Rahman, On the Enestrom-Kakeya Theorem, Tohuku Mathematics Journal, 20(1968),126-136.

[3]. M.H.Gulzar, Ajaz Wani and Rubia Akhtar, Zero-free Regions and Bounds for the Number of Zeros of a Polynomial in a Region, International Journal of Advance Research in Science and Engineering, Vol.7 Issue 3,March 2018,77-87.

M.H.Gulzar "On the Bounds for the Zeros of a Polynomial ." IOSR Journal of Engineering (IOSRJEN), vol. 08, no. 7, 2018, pp. 28-37