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Reliability Prediction of State Dependent Retrial Queue with Unreliable Server and Multi-Phase Repair

Neetu Singh
Department of Applied Sciences, World College of Technology & Management, Gurgaon (Haryana), India

Abstract: - This paper deals with a single server retrial queueing system in which customers arrive according to poisson fashion with state dependent rates. An arriving customer enters into service immediately on finding the server free; otherwise the customer enters into a retrial orbit and repeatedly attempts to access the server at independent and identically distributed intervals. We study the classical and constant retrial policy in accordance with the discipline to access the server from the orbit. The service interruption due to server breakdown is taken into consideration. The repairman repairs the server in m-phases and also requires general distributed set up time before starting repair of 1^{st} phase. The life-time and phase repair time of the server are assumed to be according exponential and general distributed, respectively. We perform the steady state analysis of the model using supplementary variable technique and Laplace transform; then employ the probability generating function approach to derive expressions for various performance measures. Finally, numerical illustration is given to explore the effect of various parameters on the system performance.

Keywords: - State dependent, Retrial queue, Supplementary variable, Server breakdown, Set up time, Multiphase repair, Reliability.

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I. INTRODUCTION

The literature on retrial queues is increasing rapidly as these models arise frequently in the performance prediction of many telecommunications, computer networks and manufacturing/production systems, etc.. Moreover, retrial queues are used to examine the queueing situations of advanced communication systems including packet switching networks, shared bus local area networks operating under the Carrier-Sense Multiple Access protocol and many others. Retrial queueing systems differ from classical queueing systems in the sense that in classical queueing systems it is assumed that the customers are in continuous contact with the server i.e. they can observe server's state, whether it is busy or not and thus starts getting service immediately whenever the service station becomes idle. On the other hand in some retrial queueing systems the customers do not know about the server's state and therefore they have to check from time to time if the server is idle.

In general the single server retrial queue can be defined as a group (orbit) of unserved customers, which is formed by the accumulation of the customers who find the server busy upon arrival, and leave service area to join the retrial orbit. In this orbit the customers wait for some time (retrial time) and conduct a repeated attempt independently of each other until they find the server free and working. Keilson et al. (1968) used the method of supplementary variable to analyze M/G/1 retrial queue. Some other important contributions in this direction are due to Yang and Templeton (1987), Falin (1990), Rege (1993), Falin and Templeton (1997), Rodrigo et al. (1998). The accessible bibliography on retrial queues can be found in Artelejo (1999). Krishana Kumar and Arivudainambi (2002) derived analytical results for some performance measures of the M/G/1 retrial system under steady state. Moreno (2004), Wenhui (2005) considered single server retrial queue with general retrial times. Retrial queue with Bernoulli schedule in different frameworks have been taken into consideration by Atentia et al. (2006), Atentia and Moreno (2005), Atencia et al. (2009) and Choudhury and Ke (2012). Atentia et al. (2006) analyzed retrial queue with classical as well as constant retrial policy. Dudin et al. (2015) considered a retrial queueing system with a single server and group admission of customers. M/G/1 feedback retrial queue server breakdown and repair under multiple working vacation policy was studied by Rajadurai et al. (2018)

In many real time systems, the server is subject to random breakdowns when it is in working state. The performance of any system is highly influenced by the server breakdown as such it is of vital importance to study performance and reliability of retrial systems with unreliable servers. Many researchers have analyzed retrial queueing system with server breakdown from time to time. Aissani (1988), Kulkarni and Choi (1990), Aissani (1993) considered the retrial queues with server subject to breakdowns. The M/G/1 retrial system in which server failure takes place before starting the service was considered by Yang and Li (1994). Artalejo

(1994), Aissani and Artalejo (1998), Wang et al. (2001) studied such system in which failures take place after a random amount of service time. Almasi et al. (2005) investigated single server retrial queue with a finite number of homogeneous calls and a single nonreliable server; the server is subject to random breakdowns depending on whether it is busy or idle. Performance and reliability of retrial queueing systems with unreliable server have been analyzed by Gharbi and Ioualalen (2006). Choudhury et al. (2010) investigate the steady-state behavior of an $M^X/G/1$ retrial queue with an additional second phase of optional service and service interruption where breakdowns occur randomly at any instant while the server is serving the customers. Zhang and Wang (2013) presented an analytic approach for investigating a single-server retrial queue with finite population of customers where the server is subject to interruptions. Chang et al. (2018) studied unreliable-server retrial queue with customer feedback and impatience.

For some real queueing systems the incorporation of the server's set up time is required, which can be seen in computer networks, production systems, airline and railway scheduling, etc.. M(n)/G/1/N queues with set up time and state dependent arrival rates was analyzed by Li et al. (1995); they developed an efficient algorithm for computing the stationary queue length distribution. *N* policy M/G/1 queueing system with setup time for different models have been taken into consideration by Hur and Paik (1999) and Wang et al. (2007). Lee and Kim (2007) considered exponentially distributed set up time for M/G/1 queueing system. Performance analysis of single-server retrial queue with Bernoulli schedule and set up has been done by Wenhui (2005) and Krishna Kumar et al. (2013). Equilibrium pricing in an M/G/1 retrial queue with reserved idle time and setup time was investigated by Zhang and Wang (2017).

In the present paper, we consider classical as well as constant retrial policy. In classical retrial policy all the customers in retrial queue attempt at a fixed retrial rate but in constant retrial policy they are discouraged and reduce their retrial rate depending upon the number of customers present in retrial orbit. The probability of repeated attempt during the interval $(t, t+dt)$, when there are n customers in the orbit at time t, is $n\theta dt + o(dt)$ for classical retrial policy whereas for constant retrial policy it is $\theta dt + o(dt)$. Fayolle (1986) first introduced the constant retrial policy. The classical as well as constant retrial policy has been studied by few researchers (cf. Atentia et al. (2006); Jain et al. (2008))

The model under consideration is an extension of study done by Wang et al. (2001) for M/G/1 retrial queue with unreliable server. We incorporate the concepts of state dependent arrival rates, general distributed setup time and repair in m-phases. The rest of the paper is organized in different sections in the following manner. The requisite assumptions and notations are described in section 2. Afterwards in section 3, the steady state equations governing the model are obtained using supplementary variable technique. Analysis of queue size distribution is established by using Laplace transforms and generating function in section 4. Section 5 provides expressions for various performance indices characterizing the system. In the next section 6, the cost analysis is done. Some special cases are discussed in section 7. Some reliability indices are obtained in section 8. The numerical illustration and sensitivity analysis is performed in section 9 and section 10, respectively. Finally, concluding remarks are given in section 11.

II. THE MODEL

In this section, we develop a model for a single server retrial queueing system with server subject to breakdown and subject to repair in phases. This model is developed under Poisson stream with state dependent rates. The basic assumptions governing the model are as follows:

- The customers arrive according to a Poisson stream with state dependent rates λ_I, λ_B, λ_S and $\lambda_j; (j = 1, 2, ..., m)$, depending upon the status of the server who may be idle, busy, under set up and j[th] (j=1,2,…, m) repair phase, respectively.
- The arriving customers examine whether the server is available and idle; on finding the server idle, the customer is attended by the server immediately, on the other hand if the server is busy or broken-down, the arriving customers join a retrial group (orbit). As the customers in orbit are not aware of the server's state, they repeat their request or service again and again after a random amount of time; the time between two successive attempts of the same customer is exponentially distributed with rate θ_n, when there are n customers in the retrial orbit at time t.
- We consider the two policies of retrial, first is classical retrial policy in which the probability of a repeated attempt during the given time interval is dependent on the number of customers present in the system and is given by $n\theta dt + o(dt)$, as such $\theta_n = \theta$. However in constant retrial policy the probability of a repeated attempt during the given time interval is $\theta dt + o(dt)$, so that the retrial rate of the repeated customers is given

by $\theta_n = \theta / n$, when there are n repeated customers in the system. As the repeated customers are discouraged when more number of customers join the orbit.

- The service times of customers are assumed to be independent and identically distributed random variables with common probability distribution function A (.), probability density function a (.). Laplace transforms and k^{th} moment of a (.) respectively are denoted by $a^*(s)$ and $\eta_k = (-1)^k a^{*(k)}(0)$, k=1,2,… , respectively.

Now $a(u) = \mu(u) \exp .\{-\int_0^u \mu(t)dt\}$, where $\mu(u)$ is service completion rate given by

$$\mu(u) = \frac{a(u)}{\overline{A}(u)} .$$

- The server is subject to breakdown and the life time of the server follows an exponential distribution with rate $\alpha > 0$.
- The repairman requires set up time before starting the repair of broken down server and renders repair of the server in m phases. The random variables v and w_j (j=1,2,…, m) denote the time required for the set up and j^{th} phase repair with common probability distribution functions B(.) and G_j (.), respectively; the corresponding probability density functions are b(.) and g_j(.), respectively. Laplace transforms of b(.) and g(.) are $b^*(s)$ and $g_j^*(s)$ whereas k^{th} moments are $\beta_k = (-1)^k b^{*(k)}(0)$ and $\zeta_k^j = (-1)^k g_j^{*(k)}(0); j = 1,2,...,m$; k=1,2,…, respectively.

Also $b(v) = \upsilon(v) \exp\{-\int_0^v v(t)dt\}$ and $g_j(w_j) = \beta_j(w_j) \exp\{-\int_0^{w_j} \beta_j(t)dt\}$, where $\upsilon(v)$ and $\beta_j(w_j)$ are set up and repair completion rates, respectively and are given by

$$\upsilon(v) = \frac{b(v)}{\overline{B}(v)} \text{ and } \beta_j(w_j) = \frac{g_j(w_j)}{\overline{G}_j(w_j)}, \quad j = 1,2,...,m .$$

We also denote $\mathbf{w_j} = (w_1, w_2, ..., w_j), \; j = 1,2,...,m$.

- $E(I), E(B), E(S), E(R_j)$ and $E(C)$ denote the expected length of idle period, busy period, set up period and j^{th} (j=1,2,…, m) phase repair period and cycle duration, respectively.

Stochastic behavior of the retrial system can be described by the Markov process $\{X(t): t \ge 0\} = \{C(t), N(t), U(t), V(t), W_j(t): t \ge 0, \; j = 1, 2, ..., m\}$, by making use of supplementary variables $U(t), V(t)$ and $W_j(t)$. Here $C(t)$ denotes the server's state at time t ; I, B, S, j, indicate that the server is idle, busy, under set up and j^{th} (j=1,2,…,m) phase of repair, respectively. Random variable N (t) is corresponding to the number of repeated customers at time t. $U(t), V(t), W_j(t)$ denote the random variables corresponding to elapsed service time, set up time and repair time for j^{th} (j=1,2,…, m) phase, respectively at time t.

The state probabilities are defined as:

$$P_{I,n}(t) = P[C(t) = I, N(t) = n]; t \ge 0, n \ge 0$$

$$P_{B,n}(t,u)du = P[C(t) = B, N(t) = n, u < U(t) < u + du]; t \ge 0, n \ge 0, u \ge 0$$

$$P_{S,n}(t,u,v)dv = P[C(t) = S, N(t) = n, U(t) = u, v < V(t) < v + dv]; t \ge 0, n \ge 0, u \ge 0, v \ge 0$$

$$P_{j,n}(t,u,v,\mathbf{w_j})dw_j = P[C(t) = j, N(t) = n, U(t) = u, V(t) = v, \; w_j < W_j(t) < w_j + dw_j] ;$$
$$t \ge 0, n \ge 0, \; u \ge 0, v \ge 0, w_j \ge 0 , \; \mathbf{w_j} = (w_1, w_2, ..., w_j), \; j = 1,2,...,m$$

The state probabilities are defined as:

$$P_{I,n} = \lim_{t\to\infty} P_{I,n}(t), \; P_{B,n}(u) = \lim_{t\to\infty} P_{B,n}(t,u), \; P_{S,n}(u,v) = \lim_{t\to\infty} P_{S,n}(t,u,v)$$

$$P_{j,n}(u,v,\mathbf{w_j}) = \lim_{t\to\infty} P_{j,n}(t,u,v,\mathbf{w_j}); \; \mathbf{w_j} = (w_1, w_2, ..., w_j), \; j = 1,2,...,m .$$

III. STEADY STATE EQUATIONS AND THEIR SOLUTION

In this section, we construct the system of the equations in steady state, governing the model by using the supplementary variables under certain initial, boundary and normalization conditions which are as follows:

$$(\lambda_I + n\theta_n)P_{I,n} = \int_0^\infty \mu(u)P_{B,n}(u)du \qquad \text{...(1.a)}$$

$$\left[\frac{\partial}{\partial u} + \lambda_B + \mu(u) + \alpha\right]P_{B,n}(u) = \int_0^\infty \beta_m(w_m)P_{m,n}(u,v,\mathbf{w_m})dw_m + \lambda_B P_{B,n-1}(u) \qquad \text{...(1.b)}$$

$$\left[\frac{\partial}{\partial v} + \lambda_S + \upsilon(v)\right]P_{S,n}(u,v) = \lambda_S P_{S,n-1}(u,v) \qquad \text{...(1.c)}$$

$$\left[\frac{\partial}{\partial w_j} + \lambda_j + \beta_j(w_j)\right]P_{j,n}(u,v,\mathbf{w_j}) = \lambda_j P_{j,n-1}(u,v,\mathbf{w_j});\ j = 1,2,...,m \qquad \text{...(1.d)}$$

The initial conditions are

$P_{I,0}(0) = 1$, $P_{I,n}(0) = 0,\ n > 0$

$P_{B,n}(0,u) = P_{S,n}(0,u,v) = P_{j,n}(0,u,v,\mathbf{w_j}) = 0$ for $n \geq 0$ and j=1,2,…,m

The boundary conditions are given by

$$P_{B,n}(0) = \lambda_I P_{I,n} + (n+1)\theta_n P_{I,n+1} \qquad \text{...(2.a)}$$

$$P_{S,n}(u,0) = \alpha P_{B,n}(u) \qquad \text{...(2.b)}$$

$$P_{1,n}(u,v,0) = \int_0^\infty \upsilon(v)P_{S,n}(u,v)dv \qquad \text{...(2.c)}$$

$$P_{j,n}(u,v,\mathbf{w_{j-1}},0) = \int_0^\infty \beta_{j-1}(w_{j-1})P_{j-1,n}(u,v,\mathbf{w_{j-1}})dw_{j-1};$$

$$\mathbf{w_{j-1}} = (w_1, w_1, ..., w_{j-1}),\ j = 2,3,...,m \qquad \text{...(2.d)}$$

The normalization condition is

$$\sum_{n=0}^{\infty}\left[P_{I,n} + \int_0^\infty P_{B,n}(u)du + \int_0^\infty\int_0^\infty P_{S,n}(u,v)dudv + \sum_{j=1}^{m}\int_0^\infty\int_0^\infty ....\int_0^\infty P_{j,n}(u,v,w_1w_2....w_j)du\,dvdw_1....dw_j\right] = 1$$

...(3)

We introduce following generating functions:

$$\left.\begin{aligned} &P_I(z) = \sum_{n=0}^{\infty} P_{I,n}z^n; \qquad P_B(z,u) = \sum_{n=0}^{\infty} P_{B,n}(u)z^n; \\ &P_S(z,u,v) = \sum_{n=0}^{\infty} P_{S,n}(u,v)z^n; \quad P_j(z,u,v,\mathbf{w_j}) = \sum_{n=0}^{\infty} P_{R,n}(u,v,\mathbf{w_j})z^n;\ |z| \leq 1 \end{aligned}\right\} \qquad \text{...(4)}$$

Multiplying Eqs. 1(b-d) and 2(b-d) by z^n and summing over n, we get

$$\left[\frac{\partial}{\partial u} + \lambda_B(1-z) + \mu(u) + \alpha\right]P_B(z,u) = \int_0^\infty \beta_m(w_m)P_m(u,v,\mathbf{w_m})dw_m \qquad \text{...(5)}$$

$$\left[\frac{\partial}{\partial v} + \lambda_S(1-z) + \upsilon(v)\right]P_S(z,u,v) = 0 \qquad \text{...(6)}$$

$$\left[\frac{\partial}{\partial w_j} + \lambda_j(1-z) + \beta(w_j)\right]P_j(z,u,v,\mathbf{w_j}) = 0,\ j = 1,2,...,m \qquad \text{...(7)}$$

and

$$P_S(z,u,0)=\alpha P_B(z,u) \qquad \ldots(8)$$
$$P_1(z,u,v,0)=\upsilon(v)P_S(z,u,v) \qquad \ldots(9)$$
$$P_j(z,u,v,\mathbf{w_{j-1}},0)=\beta_{j-1}(w_{j-1})P_{j-1}(z,u,v,\mathbf{w_{j-1}}) \quad ,\ j=2,3,...,m \qquad \ldots(10)$$

The solutions of the differential Eqs. (6)-(7) by substituting Eqs. (9) and (10) respectively, are

$$P_S(z,u,v)=\alpha P_B(z,u)e^{-\lambda_S(1-z)v}\overline{B}(v) \qquad \ldots(11)$$
$$P_1(z,u,v,w_1)=\upsilon(v)P_S(z,u,v)e^{-\lambda_1(1-z)w_1}\overline{G}(w_1) \qquad \ldots(12)$$
$$P_j(z,u,v,\mathbf{w_j})=\beta_{j-1}(w_{j-1})P_{j-1}(z,u,v,\mathbf{w_{j-1}})e^{-\lambda_j(1-z)w_j}\overline{G}(w_j),\ j=1,2,...,m \qquad \ldots(13)$$

Now Eqs. (11)-(13) lead to

$$P_m(z,u,v,\mathbf{w}_m)=\left(\alpha\upsilon(v)\prod_{i=1}^{m-1}\beta_i(w_i)\prod_{j=1}^{m}e^{-\lambda_j(1-z)w_j}\overline{G}(w_j)\right)e^{-\lambda_s(1-z)w_s}\overline{B}(v)P_B(z,u) \qquad \ldots(14)$$

From Eqs. (5), (14) and (8), we have

$$\begin{aligned}P_B(z,u)&=P_B(z,0)e^{-h(z)u}\overline{A}(u)\\ &=[\lambda_I P_I(z)+\theta P_I'(z)]e^{-h(z)u}\overline{A}(u)\end{aligned} \qquad \ldots(15)$$

where the auxiliary function h(z) is given by

$$h(z)=\lambda_B(1-z)+\alpha\left[1-\left(b^*\{\lambda_S(1-z)\}\prod_{j=1}^{m}g_j^*\{\lambda_j(1-z)\}\right)\right]$$

and satisfies the properties

(i) $h(1)=0$ (ii) $h'(1)=-\xi=\lambda_B+\alpha\left[\gamma_1\lambda_S+\sum_{j=1}^{m}\zeta_1^{(j)}\lambda_j\right]$

3.1 Classical retrial policy: In this case $\theta_n=\theta$, so that we proceed as follows:

Multiplying Eqs. 1(a) and 2(a) by z^n and summing over n, we get

$$\lambda_I P_I(z)+z\theta P_I'(z)=\int_0^\infty \mu(u)P_B(z,u)du \qquad \ldots(16)$$
$$P_B(z,0)=\lambda_I P_I(z)+\theta P_I'(z) \qquad \ldots(17)$$

Solving Eqs. (16) and (15), we obtain

$$P_I'(z)=\frac{\lambda_I[a^*\{h(z)\}-1]}{\theta[z-a^*\{h(z)\}]}P_I(z) \qquad \ldots(18)$$

The solution of Eq. (18) is

$$P_I(z)=P_I(1)\exp\left[-\frac{\lambda_I}{\theta}\int_z^1\frac{[a^*\{h(x)\}-1]}{[x-a^*\{h(x)\}]}dx\right] \qquad \ldots(19)$$

Using (17) in (18), we obtain

$$P_B(z,0)=\frac{\lambda_I(z-1)}{[z-a^*\{h(z)\}]}P_I(z) \qquad \ldots(20)$$

Applying limit $z\to 1$ in Eq. (20) and using L'Hospital's rule, we have

$$P_B(1,0)=\frac{\lambda_I}{(1-\eta_1\xi)}P_I(1) \qquad \ldots(21)$$

Similarly from Eqs. (15), (11)- (13) respectively, we find

$$P_B(1,u)=\frac{\lambda_I P_I(1)}{(1-\eta_1\xi)}\overline{A}(u) \qquad \ldots(22)$$

$$P_S(1,u,v)=\frac{\alpha\lambda_I P_I(1)}{(1-\eta_1\xi)}\bar{A}(u)\bar{B}(v) \quad \ldots(23)$$

$$P_j(1,u,v,w_1)=\frac{\upsilon(v)\alpha\lambda_I P_I(1)}{(1-\eta_1\xi)}\bar{A}(u)\bar{B}(v)\bar{G}(w_1) \quad \ldots(24)$$

$$P_j(1,u,v,\mathbf{w_j})=\frac{\prod_{i=1}^{j-1}\beta_i(w_i)\upsilon(v)\alpha\lambda_I P_I(1)}{(1-\eta_1\xi)}\bar{A}(u)\bar{B}(v)\prod_{i=1}^{j}\bar{G}(w_i);\ j=2,3,\ldots,m \quad \ldots(25)$$

Using normalization condition we find the unknown constant as

$$P_I(1)=\left(\frac{1-\xi\eta_1}{\chi}\right) \quad \ldots(26)$$

where

$$\chi=\left[1-\eta_1[\xi-\lambda_I\{1+\alpha(\gamma_1+\sum_{j=1}^{m}\zeta_1^{(j)})\}]\right]$$

Thus from (13), we obtain

$$P_I(z)=\frac{1-\xi\eta_1}{\chi}\exp\left[\frac{-\lambda_I}{\theta}\int_z^1\frac{a^*\{h(t)\}-1}{[t-a^*\{h(t)\}]}dt\right] \quad \ldots(27)$$

3.2 Constant retrial policy: In this case substituting $\theta_n=\theta/n$, the solution can be obtained as follows:

Multiplying Eqs. (1.a) and (2.a) by z^n and summing over n, we obtain

$$(\lambda_I+\theta)P_I(z)-\theta P_{0,0}=\int_0^\infty\mu(u)P_B(z,u)du \quad \ldots(28)$$

$$P_B(z,0)=\left(\lambda_I+\frac{\theta}{z}\right)P_I(z)-\frac{\theta}{z}P_{0,0} \quad \ldots(29)$$

Substitution of Eq. (15) in Eq. (28) leads to

$$(\lambda_I+\theta)P_I(z)-\theta P_{0,0}=a^*\{h(z)\}P_B(z,0) \quad \ldots(30)$$

From Eqs. (29) and (30), we obtain

$$P_I(z)=\frac{\theta[z-a^*\{h(z)\}]}{z\lambda_I[1-a^*\{h(z)\}]+\theta[z-a^*\{h(z)\}]}P_{0,0}$$

By Eq. (29), we get

$$P_B(z,0)=\frac{(z-1)\lambda_I}{[z-a^*\{h(z)\}]}P_I(z) \quad \ldots(31)$$

Proceeding similarly as in previous section and using normalizing condition, the probability that the system is free, is obtained as

$$P_{0,0}=\frac{1-\xi\eta_1\left(\frac{\lambda_I}{\theta}+1\right)}{\chi} \quad \ldots(32)$$

IV. THE GENERATING FUNCTIONS

To quantify the various measures of performance for the queueing system, the probability generating function approach can be employed. In this section we establish the probability generating function and marginal generating functions of the queue size distribution.

Eqs. (15) and (20) give

$$P_B(z,u)=\frac{\lambda_I(z-1)}{[z-a^*\{h(z)\}]}e^{-h(z)u}\bar{A}(u)P_I(z) \quad \ldots(33)$$

From Eqs. (11)-(13) with the recursive use of Eqs. (33)-(35), we get

$$P_S(z,u,v)=\frac{\alpha\lambda_I(z-1)}{[z-a^*\{h(z)\}]}e^{-h(z)u}e^{-\lambda_S(1-z)v}\bar{A}(u)\bar{B}(v)P_I(z) \qquad \ldots(34)$$

$$P_1(z,u,v,w_1)=\frac{\upsilon(v)\alpha\lambda_I(z-1)\beta_1(w_1)}{[z-a^*\{h(z)\}]}e^{-h(z)u}e^{-(\lambda_S v+\lambda_1 w_1)(1-z)}\bar{A}(u)\bar{B}(v)\bar{G}(w_1)P_I(z) \qquad \ldots(35)$$

$$P_j(z,u,v,\mathbf{w_j})=\frac{\upsilon(v)\alpha\lambda_I(z-1)\prod_{i=1}^{j-1}\beta_j(w_j)}{[z-a^*\{h(z)\}]}e^{[-h(z)u]}e^{-[\lambda_S v+\sum_{j=1}^{m}\lambda_j w_j](1-z)}\bar{A}(u)\bar{B}(v)\prod_{i=1}^{j}\bar{G}(w_i)P_I(z)$$

$$j=1,2,\ldots,m \ldots(36)$$

Eqs. (33)-(36) represent the partial probability generating functions of queue size when the server's states are busy, set up, and j[th] phase of repair, respectively.

Using Eqs. (33)- (36) and the result $\int_0^\infty e^{-su}[1-\bar{A}(u)]du=\frac{\bar{a}^*(s)}{s}$,

we get the marginal generating functions as follows:

$$P_B(z)=\int_0^\infty P_B(z,u)du=\frac{\lambda_I(z-1)[\bar{a}^*\{h(z)\}]}{h(z)[z-a^*\{h(z)\}]}P_I(z) \qquad \ldots(37)$$

$$P_S(z)=\int_0^\infty\int_0^\infty P_S(z,u,v)dudv$$

$$=\frac{-\alpha\lambda_I[\bar{a}^*\{h(z)\}][\bar{b}^*\{\lambda_S(1-z)\}]}{\lambda_S h(z)[z-a^*\{h(z)\}]}P_I(z) \qquad \ldots(38)$$

$$P_1(z)=\int_0^\infty\int_0^\infty\int_0^\infty P_1(z,u,v,w_1)dudvdw_1$$

$$=\frac{-\alpha\lambda_I[\bar{a}^*\{h(z)\}][b^*\{\lambda_S(1-z)\}][\bar{g_1}^*\{\lambda_1(1-z)\}]}{\lambda_S h(z)[z-a^*\{h(z)\}]}P_I(z) \qquad \ldots(39)$$

$$P_j(z)=\int_0^\infty\int_0^\infty\ldots\int_0^\infty P_1(z,u,v,w_1)dudvdw_1\ldots dw_j$$

$$=\frac{-\alpha\lambda_I[\bar{a}^*\{h(z)\}][b^*\{\lambda_S(1-z)\}][\bar{g}^*_j\{\lambda_j(1-z)\}\prod_{i=1}^{j-1}[g_1^*\{\lambda_1(1-z)\}]}{\lambda_S h(z)[z-a^*\{h(z)\}]}P_I(z)$$

$$,\ j=1,2,\ldots,m \qquad \ldots(40)$$

Probability generating function of the number of repeated customers is

$$H_1(z)=P_I(z)+P_B(z)+P_S(z)+\sum_{j=1}^{m}P_j(z)$$

$$=\frac{P_I(z)}{h(z)[z-a^*\{h(z)\}]}\Bigg[h(z)[z-a^*\{h(z)\}]+\lambda_I(z-1)\bar{a}^*h(z)-\frac{\alpha\lambda_I}{\lambda_S}\bar{a}^*h(z)\bar{b}^*\{\lambda_S(1-z)\}$$

$$-\alpha\lambda_I\bar{a}^*h(z)\bar{b}^*\{\lambda_S(1-z)\}\sum_{j=2}^{m}\frac{1}{\lambda_j}\prod_{i=1}^{j}\bar{g_i}^*\{\lambda_i(1-z)\}\Bigg] \qquad \ldots(41)$$

Probability generating function of the number of customers in the system is

$$H_2(z) = P_I(z) + zP_B(z) + zP_S(z) + z\sum_{j=1}^{m} P_j(z)$$

$$= \frac{P_I(z)}{h(z)[z - a^*\{h(z)\}]}\Bigg[h(z)[z - a^*\{h(z)\}] + z\lambda_I(z-1)\bar{a}^* h(z) - \frac{\alpha\lambda_I}{\lambda_S} z\bar{a}^* h(z)\bar{b}^*\{\lambda_S(1-z)\}$$

$$-\alpha\lambda_I z\bar{a}^* h(z)\bar{b}^*\{\lambda_S(1-z)\}\sum_{j=2}^{m}\frac{1}{\lambda_j}\prod_{i=1}^{j}\bar{g}_i^*\{\lambda_i(1-z)\}\Bigg] \qquad \ldots(42)$$

V. PERFORMANCE METRICES

Performance metrices play an important role to predict the performance of the system. In this section, we provide expressions for various system characteristics using which the performance of the system can be predicted. Throughout this section, it is assumed that retrial policy is classical. The results for constant retrial policy may be derived similarly.

5.1 Probability of server's state:

When the system is in steady state then the probabilities for different states are derived using marginal probability generating functions as follows:

The probability of the server being idle is

$$P(I) = \lim_{z\to 1} P_I(z) = \frac{1-\xi\eta_1}{\chi} \qquad \ldots(43)$$

The probability of the server being busy is given by

$$P(B) = \lim_{z\to 1}\int_0^\infty P_B(z,u)du = \frac{\lambda_I\eta_1}{\chi} \qquad \ldots(44)$$

The probability that the server is in set up state, is obtained using

$$P(S) = \lim_{z\to 1}\int_0^\infty\int_0^\infty P_S(z,u,v)dudv = \frac{\alpha\lambda_I\eta_1\gamma_1}{\chi} \qquad \ldots(45)$$

The probability that the server is in j^{th} (j=1,2,…,m) phase of repair, is determined by

$$P(R_j) = \lim_{z\to 1}\int_0^\infty\int_0^\infty\ldots\int_0^\infty P_j(z,u,v,w_1,w_2,\ldots,w_j)dudvdw_1dw_2\ldots.dw_j = \frac{\alpha\lambda_I\eta_1\zeta_1^j}{\chi}$$

…(46)

5.2 Average queue size:

Now we find expressions for the expected number of customers in the retrial queue and in the system in case of 2-phase repair as follows:

The expected number of customers in the retrial queue is given by

$$E(L_1) = \lim_{z\to 1} H_1(z)$$

$$= \frac{\lambda_I}{\theta}\left(\frac{\eta_1\xi}{1-\eta_1\xi}\right) + \frac{1}{2\chi\xi}\left(\alpha\lambda_I\eta_1\xi\varepsilon_2 - \psi\eta_1\varepsilon_1 + \frac{(\eta_2\xi^2 + \eta_1\varepsilon_1)\psi}{(1-\eta_1\xi)}\right) \qquad \ldots(47)$$

where

$$\varepsilon_1 = \alpha[\lambda_S^2\gamma_2 + \lambda_1^2\zeta_2^{(1)} + \lambda_2^2\zeta_2^{(2)} + 2\lambda_S\lambda_1\zeta_1^{(1)}\gamma_1 + 2\lambda_S\lambda_2\zeta_1^{(2)}\gamma_1 + 2\lambda_1\lambda_2\zeta_1^{(1)}\zeta_1^{(2)}]$$

$$\varepsilon_2 = [\lambda_S\gamma_2 + \lambda_1\zeta_2^{(1)} + \lambda_2\zeta_2^{(2)} + 2\lambda_S\gamma_1\zeta_1^{(1)} + 2\lambda_S\gamma_1\zeta_1^{(2)} + 2\lambda_1\zeta_1^{(1)}\zeta_1^{(2)}]$$

$$\psi = \lambda_I[1 + \alpha(\gamma_1 + \zeta_1^{(1)} + \zeta_1^{(2)})]$$

The expected number of customers in the system is obtained as

$$E(L_2) = \lim_{z\to 1} H_2(z) = E(L_1) + \frac{\eta_1\psi}{\chi} \qquad \ldots(48)$$

VI. COST FUNCTION

Customers' encounter with waiting lines can significantly affect their overall level of satisfaction with the organization. Aim of an organization is to offer sufficiently fast service, within cost constraints so that they can satisfy their customers. Before establishing the expression for the expected total cost, we define some cost elements corresponding to waiting customers and server's states as follows:

C_o =Start-up cost per unit time

C_h =Holding cost per unit time per customer in the system

C_I =Cost per unit time when server is idle

C_B =Cost per unit time when server is busy in rendering service

C_S =Cost per unit time when server is in set up state

C_j =Cost incurred per unit time when server is in j^{th} (j=1,2,…, m) phase of repair

The expected length of the cycle is given by

$$E(C) = P(I) + P(B) + P(S) + \sum_{j=1}^{m} P(R_j)$$

where

$$P(I) = \frac{E(I)}{E(C)},\ P(B) = \frac{E(B)}{E(C)},\ \ P(S) = \frac{E(S)}{E(C)},\ \ \ P(R_j) = \frac{E(R_j)}{E(C)};\ j = 1,2,...,m$$

We note that $E(B)$ can be determined using

$$E(B) = \frac{\beta_1 \chi}{1 - \xi\eta_1}$$

The expected total cost $E(T)$ per unit time is given by

$$E(T) = C_o \times \frac{1}{E(C)} + C_h E(L_2) + C_I P(I) + C_B P(B) + C_S P(S) + \sum_{j=1}^{m} C_j P(R_j) \qquad \ldots(49)$$

VII. SOME SPECIAL CASES

In this section we deduce analytical results of some special cases by setting appropriate parameters. For the cases I-III we fix the second moment of service time for some specific distributions to discuss some particular cases of our model.

Case I: M/E$_k$/1 retrial queueing model with unreliable server, set up and repair in phases:

In this case we have $\eta_1 = \dfrac{1}{\mu},\ \eta_2 = \dfrac{(k+1)}{k\mu^2}$

Now from Eqs. (45) and (46), we get

$$E(L_1) = \frac{\lambda_I}{\theta}\left(\frac{\xi}{\mu - \xi}\right) + \frac{1}{2\mu\chi\xi}\left[\alpha\lambda_I \xi\varepsilon_2 - \psi\varepsilon_1 + \frac{\{(k+1)\xi^2 + k\mu\varepsilon_1\}\psi}{k\mu(\mu - \xi)}\right] \qquad \ldots(50)$$

$$E(L_2) = E(L_1) + \frac{\psi}{\mu\chi} \qquad \ldots(51)$$

Case II: M/γ/1 retrial queueing model with unreliable server and repair in phases:

For this case substituting $\eta_1 = \dfrac{k}{\mu}, \eta_2 = \dfrac{k(k+1)}{\mu^2}$ and Eqs. (45) and (46) reduce to

$$E(L_1) = \frac{\lambda_I}{\theta}\left(\frac{k\xi}{\mu - k\xi}\right) + \frac{k}{2\mu\chi\xi}\left[\alpha\lambda_I \xi\varepsilon_2 - \psi\varepsilon_1 + \frac{k\{(k+1)\xi^2 + \mu\varepsilon_1\}\psi}{\mu(\mu - k\xi)}\right] \qquad \ldots(52)$$

$$E(L_2) = E(L_1) + \frac{k\psi}{\mu\chi} \qquad \ldots(53)$$

Case III: M/D/1 retrial queueing model with unreliable server and repair in phases:

When service time distribution is deterministic we have $\eta_1 = \dfrac{1}{\mu}, \eta_2 = (\eta_1)^2$ so that Eqs. (45) and (46) convert to

$$E(L_1) = \frac{\lambda_I}{\theta}\left(\frac{\xi}{\mu-\xi}\right) + \frac{1}{2\mu\chi\xi}\left[\alpha\lambda_I\xi\varepsilon_2 - \psi\varepsilon_1 + \frac{(\xi^2+\mu\varepsilon_1)\psi}{\mu(\mu-\xi)}\right] \quad \ldots(54)$$

$$E(L_2) = E(L_1) + \frac{\psi}{\mu\chi} \quad \ldots(55)$$

Case IV: M/G/1 retrial queueing model with unreliable server:

For this case, we have $\lambda_I = \lambda_B = \lambda_S = \lambda_j = 0$, $j = 1, 2, ..., m$. Also there is no set up and single phase repair so that we obtain

$$E(L_1) = \frac{\lambda}{\theta}\left(\frac{\eta_1\lambda(1+\alpha\zeta_1^{(1)})}{1-\eta_1\lambda(1+\alpha\zeta_1^{(1)})}\right) + \left(\frac{\eta_1\alpha\lambda^2\zeta_2^{(1)} + \eta_2\lambda(1+\alpha\zeta_1^{(1)})}{2[1-\eta_1\lambda(1+\alpha\zeta_1^{(1)})]}\right) \quad \ldots(56)$$

$$E(L_2) = E(L_1) + \eta_1\lambda(1+\alpha\zeta_1^{(1)}) \quad \ldots(57)$$

Case V: M/G/1 retrial queueing model with reliable server:

For this case, we have $\lambda_I = \lambda_B = \lambda_S = \lambda_j = 0$, $j = 1, 2, ..., m$ and $\alpha = 0$, so that Eqs. (45) and (46) reduce to

$$E(L_1) = \frac{\lambda}{\theta}\left(\frac{\eta_1\lambda}{1-\eta_1\lambda}\right) + \left(\frac{\eta_1\lambda^2}{(1-\eta_1\lambda)}\right) \quad \ldots(58)$$

$$E(L_2) = E(L_1) + \eta_1\lambda \quad \ldots(59)$$

VIII. RELIABILITY INDICES

In this section, we provide expressions for some reliability indices by considering the failure states of the server as absorbing states. For reliability analysis we construct the following transient state differential difference equations:

$$\left[\frac{d}{dt} + \lambda_I + n\theta\right]P_{I,n}(t) = \int_0^\infty \mu(u)P_{B,n}(u)du \quad \ldots(60.a)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial u} + \lambda_B + \mu(u) + \alpha\right]P_{B,n}(t,u) = \lambda_B P_{B,n-1}(t,u) \quad \ldots(60.b)$$

$$P_{B,n}(t,0) = \lambda_I P_{I,n}(t) + (n+1)\theta P_{I,n+1}(t) \quad \ldots(60.c)$$

with initial condition

$P_{I,0}(0) = 1$, $P_{I,n}(0) = 0, n > 0$.

Taking Laplace transform of both sides of Eqs. (60.a)-(60.c), we get

$$sP_{I,n}^*(s) - P_{I,0}(0) = -(\lambda_I + n\theta)P_{I,n}^*(s) + \int_0^\infty \mu(u)P_B^*(s,u)du \quad \ldots(61.a)$$

$$sP_{B,n}^*(s,u) + \frac{\partial}{\partial u}P_{B,n}^*(s,u) = -[\lambda_B + \alpha + \mu(u)]P_{B,n}^*(s,u) + \lambda_B P_{B,n-1}^*(s,u) \quad \ldots(61.b)$$

$$P_{B,n}^*(s,0) = \lambda_I P_{I,n}^*(s) + (n+1)\theta P_{I,n+1}^*(s) \quad \ldots(61.c)$$

Probability generating functions in terms of Laplace transforms are defined as

$$P_I^*(s,z) = \sum_{n=0}^\infty P_{I,n}^*(s)z^n, \; P_B^*(s,z,u) = \sum_{n=0}^\infty P_{B,n}^*(s,u)z^n \quad \ldots(62)$$

Multiplying Eqs. (61.a)-(61.c) by z^n and summing over n, we get

$$(s+\lambda_I)P_I^*(s,z)-1=-z\theta\frac{\partial}{\partial z}P_I^*(s,z)+\int_0^\infty \mu(u)P_B^*(s,z,u)du \qquad \text{…(63.a)}$$

$$sP_B^*(s,z,u)+\frac{\partial}{\partial u}P_B^*(s,z,u)=-[\lambda_B+\alpha+\mu(u)]P_B^*(s,z,u)+\lambda_B zP_B^*(s,z,u) \qquad \text{…(63.b)}$$

$$P_B^*(s,z,0)=\lambda_I P_I^*(s,z)+\theta\frac{\partial}{\partial z}P_I^*(s,z) \qquad \text{…(63.c)}$$

Solving Eq. (63.b) and using Eq. (63.c), we get

$$P_B^*(s,z,u)=\left[\lambda_I P_I^*(s,z)+\theta\frac{\partial}{\partial z}P_I^*(s,z)\right]\exp[-\{s+\alpha+\lambda_B(1-z)\}]u\bar{A}(u) \qquad \text{…(64)}$$

$$\theta\left[b^*\{s+\alpha+\lambda_B(1-z)\}-z\right]\frac{\partial}{\partial z}P_I^*(s,z)=\left[s+\lambda_I-\lambda_I b^*\{s+\alpha+\lambda_B(1-z)\}\right]P_I^*(s,z)-1 \qquad \text{…(65)}$$

let $g(z)=b^*\{s+\alpha+\lambda_B(1-z)\}-z$ …(66)

We observe that

$$g(0)=b^*\{s+\alpha+\lambda_B\}\geq 0,\ g(1)=b^*\{s+\alpha\}-1<0$$

and $g''(z)=\lambda_B^2 b^{*''}\{s+\alpha+\lambda_B(1-z)\}\geq 0$

It is noticed that g (z) has exactly one root (say r) in the interval [0,1], also g (z) is strictly positive in the interval $0\leq z<r$ and strictly negative in the interval $r<z<1$. Thus when $z\neq r$, we have

$$P_I^*(s,z)=\int_z^r \frac{1}{\theta[b^*\{s+\alpha+\lambda_B(1-r_1)\}-r_1]}\exp\left[\frac{1}{\theta}\int_{r'}^z \frac{x(s+\lambda_I)-\lambda_I b^*\{s+\alpha+\lambda_B(1-x)\}}{x[b^*\{s+\alpha+\lambda_B(1-x)\}]}dx\right]dr_1 \qquad \text{…(67)}$$

In case when $z=r$, we note that

$$P_I^*(s,r)=\frac{1}{[s+\lambda_B(1-r)]} \qquad \text{…(68)}$$

$$P_B^*(s,z)=\left[\frac{s-\lambda_B(1-z)}{[z-b^*\{s+\alpha+\lambda_B(1-r_1)\}]}P_I^*(s,z)+\frac{1}{z-b^*\{s+\alpha+\lambda_B(1-z)\}}\times\frac{\bar{b}^*\{s+\alpha+\lambda_B(1-z)\}}{\{s+\alpha+\lambda_B(1-z)\}}\right] \qquad \text{…(69)}$$

Now we shall establish some reliability indices as follows:

(i) Reliability of server in terms of Laplace transform is given by

$$R^*(s)=[P_I^*(s,z)+P_B^*(s,z)]_{z=1}$$

$$=[P_I^*(s,1)+P_B^*(s,1)]=\frac{1}{s+\alpha}+\frac{\alpha}{s+\alpha}\int_1^r \frac{1}{\theta[b^*\{s+\alpha+\lambda_B(1-r_1)\}-r_1]}\times \exp\left[\frac{1}{\theta}\int_{r'}^1 \frac{x(s+\lambda_I)-\lambda_I b^*\{s+\alpha+\lambda_B(1-x)\}}{x[b^*\{s+\alpha+\lambda_B(1-x)\}-x]}dx\right]dr_1 \qquad \text{…(70)}$$

(ii) The availability of server in steady state is obtained using

$$A=\sum_{n=0}^\infty P_{I,n}+\sum_{n=0}^\infty\int_0^\infty P_{B,n}(u)du$$

$$=\lim_{z\to 1}\left[\sum_{n=0}^\infty P_{I,n}+\sum_{n=0}^\infty P_{B,n}(u)du\right]=\frac{1-\eta_1(\xi-\lambda_I)}{\chi} \qquad \text{…(71)}$$

(iii) The failure frequency of the server in steady state is obtained as

$$F=\sum_{n=0}^{\infty}\int_0^{\infty}\alpha P_{B,n}(u)du$$

$$=\lim_{z\to 1}\int_0^{\infty}\alpha P_{B,n}(u)du=\frac{\alpha\lambda_I\eta_1}{\chi} \qquad \ldots(72)$$

(iv) The mean time to system failure (MTSF) is given by

$$MTSF=\int_0^{\infty}R(t)dt=[R^*(s)]_{s=0}$$

$$=\frac{1}{\alpha}+\int_1^{r}\frac{1}{\theta[b^*\{\alpha+\lambda_B(1-r_1)\}-r_1]}\exp\left[\frac{1}{\theta}\int_{r'}^{1}\frac{x\lambda_I-\lambda_I b^*\{\alpha+\lambda_B(1-x)\}}{x[b^*\{\alpha+\lambda_B(1-x)\}-x]}dx\right]dr_1 \qquad \ldots(73)$$

IX. SENSITIVITY ANALYSIS

Set	λ_I	λ_B	λ_S	λ_1	λ_2
A.1	2	1.9	1.8	1.7	1.6
A.2	2.5	2.4	2.3	2.2	2.1
A.3	3	2.9	2.8	2.7	2.6
A.4	3.5	3.4	3.3	3.2	3.1

Table 1: Arrival rates for different sets

Sensitivity analysis helps to find how the uncertainty in the output of a mathematical model or system can be dispensed to different sources of uncertainty in its inputs. It is also helpful in testing the robustness of the results of a model or system in the presence of uncertainty. We perform numerical experiment to examine the effect of different input parameters on various performance measures by assuming m=2, k-Erlang distribution for service time and classical retrial policy. All the computational works have been done by using software MATLAB and program is run on Pentium IV. In table 1 we have displayed different sets of arrival rates taken for figures 1 and 2. In tables 2-7, we summarize numerical results of various system characteristics by varying different parameters. The results for the expected number of customers in the retrial queue and in the system with the variation of various system parameters have been displayed in figs. 1-4.

In Tables 2-7 we set $\alpha=1/10$, $\theta=2$, $\mu=10$, $k=4$, $\lambda_I=\lambda$, $\lambda_B=\lambda$, $\lambda_S=\lambda$, $\lambda_1=\lambda$, $\lambda_2=\lambda$ and assume that set up time and repair times in 1st and 2nd phases are exponential distributed with parameters $\upsilon=15$, $\zeta^{(1)}=10$, $\zeta^{(2)}=8$, respectively.

From tables 2-4 it is noted that the probability of server being in idle state decreases with λ,α and increases with increasing values of μ whereas the probabilities of server being in busy state, set up state, 1st phase of repair state, 2nd phase of repair state increase with the increment in the values of λ and α.

In tables 5-7, we depict results for availability (failure frequency) by varying λ for different values of breakdown rate (α), service rate of server (μ) and various sets of $(\zeta^{(1)}, \zeta^{(2)})$, respectively. From table 5 we note that as λ increases, the availability (failure frequency) decreases (increases), but for fixed values of λ it also decreases (increases) with α. Similar effects are observed for different values of λ & μ, λ & $(\zeta^{(1)}, \zeta^{(2)})$ respectively from tables 6 and 7. It is noted that the availability (failure frequency) increases (decreases) with the service rate of the customers (μ) and the repair rates of the server $(\zeta^{(1)}, \zeta^{(2)})$; but decreases (increases) with the arrival rate of the customers and failure rate of the server as also seen in table 5.

In figs. 1 and 2, we depict the effect of parameters θ,α and μ on the expected number of customers in the retrial queue and in the system for different sets of arrival rates (see table 1) by setting default parameters as $\alpha=1/10$, $\theta=2$, $\mu=10, k=4$, $(\zeta_1^{(1)},\zeta_1^{(2)})=(1/20,1/15)$, $(\zeta_2^{(1)},\zeta_2^{(2)})=(1/15,1/12)$, (γ_1,γ_2)= $(1/50,1/40)$. We observe from all figures that the expected number of customers $E(L_1)$ and $E(L_2)$ increase with the increase in the arrival rates as expected. The effects of change in service rate on the expected number of customers $E(L_1)$ and $E(L_2)$ are visualized in fig. 1 (a) and 2 (a), respectively. It is seen that the increment in the values of μ results in the decrement in $E(L_1)$ and $E(L_2)$; this decrement is very sharp for smaller values of μ, whereas for higher values of μ it is almost constant. This is what we expect in physical situation that is if the service rate is higher the queue will be of smaller length, but ultimately effect diminishes. When we explore the effect of α on $E(L_1)$ and $E(L_2)$ in figs. 1(b) and 2 (b), a positive correlation between α and $E(L_1)$, and also between α and $E(L_2)$ are noticed with the increasing values of α; both $E(L_1)$ and $E(L_2)$ increase, however the effect is more prominent for higher values of α. From figs. 1 (c) and 2 (c) we observe the variation in $E(L_1)$ and $E(L_2)$ with the increase in retrial rate (θ). It is found that initially, for smaller values of θ there is a rapid decrement in $E(L_1)$ and $E(L_2)$ but for higher values of θ it decreases gradually.

For graphs in figures 3 (a)-(c) and 4 (a)-(c) we take $\lambda_I = \lambda, \ \lambda_B = \lambda, \ \lambda_S = \lambda, \ \lambda_1 = \lambda, \ \lambda_2 = \lambda$ and visualize the effect of λ on the expected number of customers in the retrial queue and in the system for different values of μ, θ and α. We observe the similar effects as seen for figures 1 and 2.

From the tables and graphs, we conclude that

- Expected number of customers in the retrial queue and in the system both increase with the increasing values of arrival rate and failure rate of the server and decreases with the increment in the values of service rate and retrial rate of customers; which are in agreement with the physical situation.
- The availability (failure frequency) of the server decreases (increases) with the increment in the arrival rate of customers and failure rate of the server whereas it increases (decreases) with the increasing values of repair rate and service rate of the server; such patterns also tally with real life observations.
- Expected total cost increases with arrival rate of customers and failure rate of server, however it can be minimized by increasing the service rate to a certain extent.

X. CONCLUSION

Retrial queues are widely used to model problems in telephone switching systems, telecommunication and computer networks and many other congestion situations. The M/G/1 retrial queue by including (i) unreliable server (ii) state dependent arrival rates (iii) setup time and (iv) repair in m-phases, is studied using supplementary variable technique and Laplace transform to establish the queue size distribution. The model is more realistic as it deals with the more versatile congestion situations than studied in earlier existing literature on retrial queues. The incorporation of state dependent rates makes our model more closer to practical problems as arrivals may be influenced by the states of the server. The concept of server breakdown together with set up time is of great interest in many real life congestion systems. Also the repair facility in phases can assist the system engineers and decision makers in improving the reliability and availability of the concerned system.

Expressions for various performance as well as reliability indices are provided in explicit form, which can be easily used to prepare ready reckners as shown by taking numerical illustration. The cost analysis done may be helpful to decision makers and practioners for smooth and reliable functioning of the system at optimum cost.

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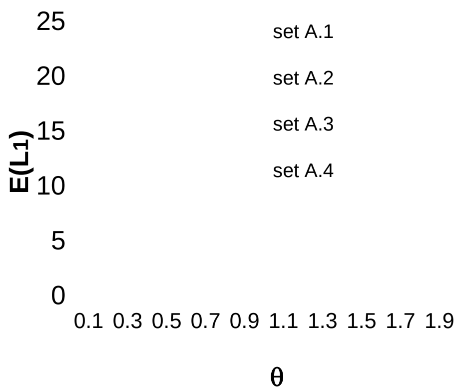

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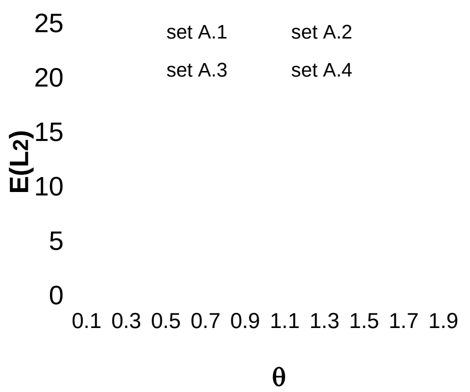

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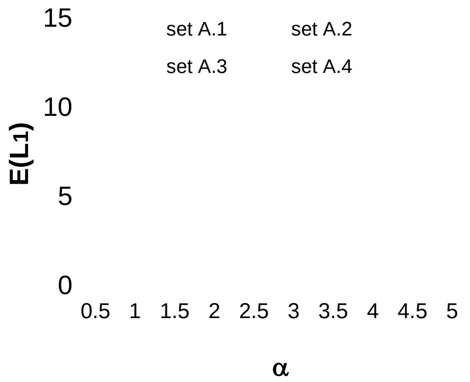

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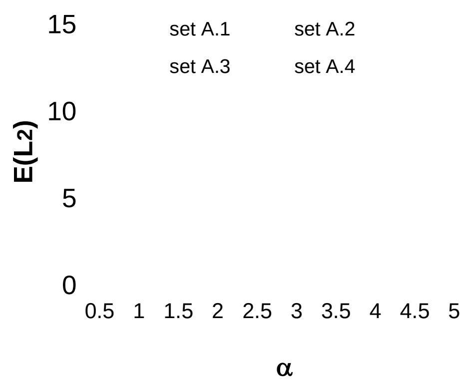

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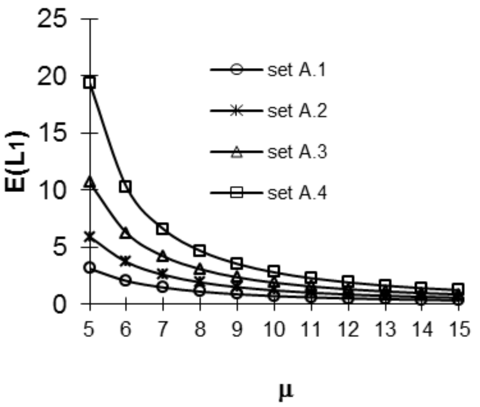

(c)

Fig. 1: $E(L_1)$ for different sets of arrival rates and (a) θ , (b) α , (c) μ

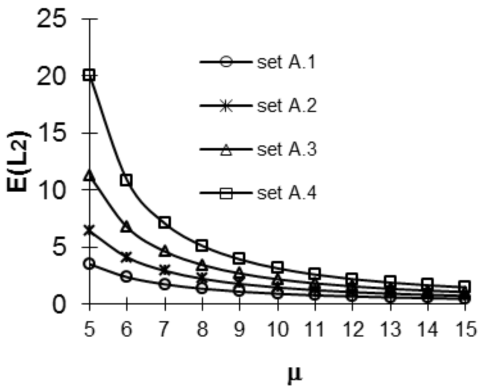

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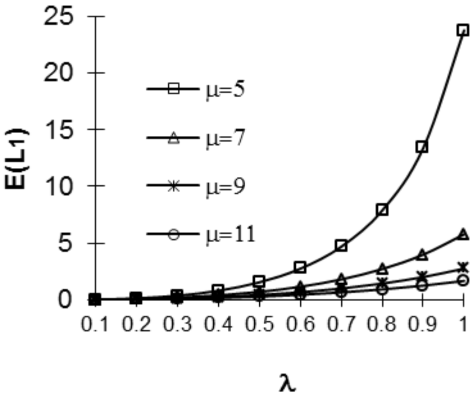
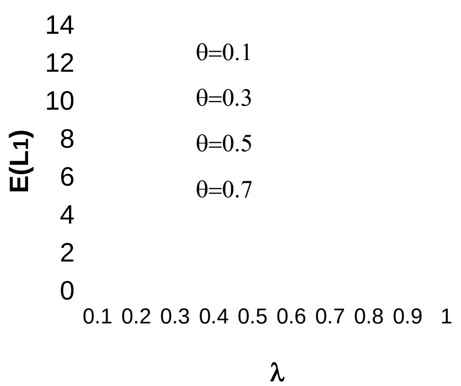
Fig. 2: $E(L_2)$ for different sets of arrival rates and (a) θ , (b) α , (c) μ



(a)



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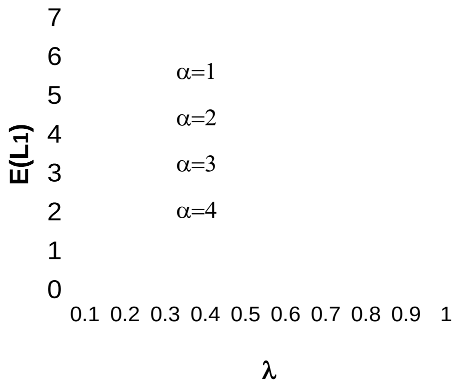
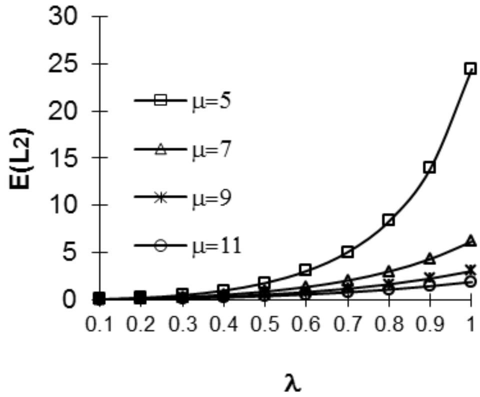


(c)

Fig. 3: Effect of λ on E(L_1) by varying (a) μ (b) θ and (c) α



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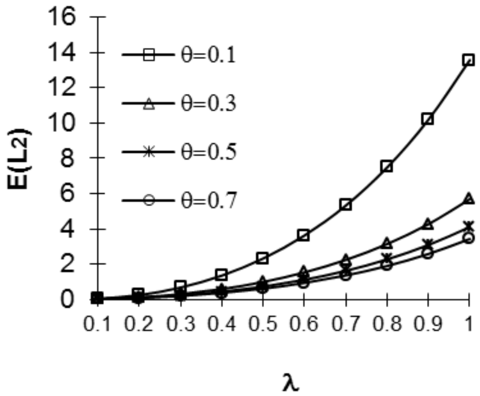

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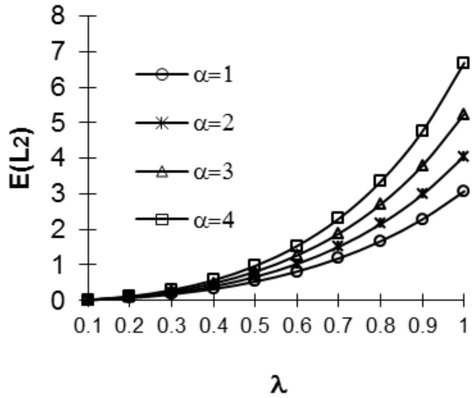

(c)

Fig. 4: Effect of λ on E(L_2) by varying (a) μ (b) θ and (c) α

λ	P(I)	P(B)	P(S)	P(RI)	P(R2)	E(TC)
1	0.8853	0.1113	0.0009	0.0011	0.0014	148.38
1.1	0.8724	0.1238	0.0010	0.0012	0.0015	161.86
1.2	0.8591	0.1366	0.0011	0.0014	0.0017	175.59
1.3	0.8456	0.1498	0.0012	0.0015	0.0019	189.60
1.4	0.8318	0.1632	0.0014	0.0016	0.0020	203.91
1.5	0.8176	0.1769	0.0015	0.0018	0.0022	218.54
1.6	0.8031	0.1910	0.0016	0.0019	0.0024	233.53
1.7	0.7882	0.2054	0.0017	0.0021	0.0026	248.91
1.8	0.7730	0.2202	0.0018	0.0022	0.0028	264.71
1.9	0.7574	0.2354	0.0020	0.0024	0.0029	280.98
2	0.7413	0.2509	0.0021	0.0025	0.0031	297.78

Table 2: Some performance measures for different values of λ

μ	P(I)	P(B)	P(S)	P(RI)	P(R2)	E(TC)
5	0.7413	0.2509	0.0021	0.0025	0.0031	197.71
6	0.7932	0.2006	0.0017	0.0020	0.0025	176.57
7	0.8278	0.1671	0.0014	0.0017	0.0021	164.76
8	0.8524	0.1432	0.0012	0.0014	0.0018	157.28
9	0.8709	0.1252	0.0010	0.0013	0.0016	152.13
10	0.8853	0.1113	0.0009	0.0011	0.0014	148.38
11	0.8968	0.1001	0.0008	0.0010	0.0013	145.53
12	0.9062	0.0910	0.0008	0.0009	0.0011	143.30
13	0.9140	0.0834	0.0007	0.0008	0.0010	141.50
14	0.9206	0.0770	0.0006	0.0008	0.0010	140.02
15	0.9263	0.0715	0.0006	0.0007	0.0009	138.78

Table 3: Some performance measures for different values μ

α	P(I)	P(B)	P(S)	P(RI)	P(R2)	E(TC)
0	0.8889	0.1111	0.0000	0.0000	0.0000	147.93
0.5	0.8707	0.1120	0.0047	0.0056	0.0070	150.31
1	0.8522	0.1130	0.0094	0.0113	0.0141	153.04
1.5	0.8334	0.1139	0.0142	0.0171	0.0214	156.18
2	0.8142	0.1149	0.0192	0.0230	0.0287	159.76
2.5	0.7948	0.1159	0.0241	0.0290	0.0362	163.83
3	0.7750	0.1169	0.0292	0.0351	0.0438	168.46
3.5	0.7548	0.1179	0.0344	0.0413	0.0516	173.71
4	0.7343	0.1190	0.0397	0.0476	0.0595	179.67
4.5	0.7135	0.1200	0.0450	0.0540	0.0675	186.43
5	0.6922	0.1211	0.0505	0.0605	0.0757	194.09

Table 4: Some performance measures for different values of α

λ	α=1		α=2		α=3	
	A	F	A	F	A	F
0.0	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.2	0.9702	0.0205	0.9607	0.0411	0.9512	0.0618
0.4	0.9389	0.0419	0.9193	0.0844	0.8994	0.1274
0.6	0.9061	0.0644	0.8755	0.1301	0.8443	0.1971
0.8	0.8717	0.0881	0.8292	0.1785	0.7856	0.2714
1.0	0.8354	0.1130	0.7802	0.2298	0.7230	0.3507
1.2	0.7973	0.1392	0.7281	0.2842	0.6559	0.4355
1.4	0.7570	0.1668	0.6727	0.3421	0.5841	0.5265
1.6	0.7145	0.1960	0.6137	0.4038	0.5068	0.6244
1.8	0.6695	0.2269	0.5507	0.4696	0.4234	0.7298
2.0	0.6218	0.2596	0.4833	0.5401	0.3333	0.8439

Table 5: Availability and failure frequency for different values of λ and α

λ	μ=5		μ=10		μ=15	
	A	F	A	F	A	F
0.0	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.2	0.9564	0.0042	0.9787	0.0020	0.9859	0.0014
0.4	0.9090	0.0087	0.9564	0.0042	0.9713	0.0027
0.6	0.8571	0.0137	0.9332	0.0064	0.9564	0.0042
0.8	0.8003	0.0191	0.9090	0.0087	0.9410	0.0056
1.0	0.7376	0.0251	0.8836	0.0111	0.9252	0.0072
1.2	0.6682	0.0317	0.8571	0.0137	0.9090	0.0087
1.4	0.5910	0.0391	0.8294	0.0163	0.8922	0.0103
1.6	0.5045	0.0474	0.8003	0.0191	0.8749	0.0120
1.8	0.4069	0.0567	0.7697	0.0220	0.8571	0.0137
2.0	0.2959	0.0673	0.7376	0.0251	0.8388	0.0154

Table 6: Availability and failure frequency for different values of λ and μ

λ	$(\zeta^{(1)}, \zeta^{(2)})$ =(5,4)		$(\zeta^{(1)}, \zeta^{(2)})$ =(15,10)		$(\zeta^{(1)}, \zeta^{(2)})$ =(25,20)	
	A	F	A	F	A	F
0.0	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
0.2	0.9476	0.0412	0.9641	0.0410	0.9685	0.0410
0.4	0.8919	0.0850	0.9263	0.0842	0.9355	0.0840
0.6	0.8326	0.1317	0.8865	0.1297	0.9008	0.1292
0.8	0.7693	0.1814	0.8444	0.1778	0.8643	0.1768
1.0	0.7016	0.2347	0.8000	0.2286	0.8258	0.2270
1.2	0.6291	0.2917	0.7529	0.2824	0.7851	0.2799
1.4	0.5511	0.3530	0.7030	0.3394	0.7421	0.3359
1.6	0.4672	0.4190	0.6500	0.4000	0.6967	0.3952
1.8	0.3765	0.4903	0.5935	0.4645	0.6484	0.4580
2.0	0.2781	0.5676	0.5333	0.5333	0.5972	0.5248

Table 7: Availability and failure frequency for different values of λ and $(\zeta^{(1)}, \zeta^{(2)})$