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# Imaginary Root Systems of Super Hyperbolic GKM Algebras $SHGGH_n^{(3)},\ n = 96,97,98$

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**Abstract:** In this paper, the Super hyperbolic GKM algebras $SHGGH_n^{(3)}$ , n=96,97,98 is studied. For this family $SHGGH_n^{(3)}$ , n=96,97,98 the existence and non-existance of purely imaginary, strictly imaginary and special imaginary roots are discussed in detail. The properties of roots were also delimited separately for each case (i.e) for n=96,97,98 respectively. A complete classification of Dynkin diagram is tabulated and the number of existing non-isomorphic connected Dynkin diagrams for the super hyperbolic family $SHGGH_n^{(3)}$ , n=96,97,98 is also given.

**Keywords:** Dynkin Diagram, Generalized Generalized Cartan Matrix, Imaginary Roots, Root System, Isotropic, Super Hyperbolic

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## I. INTRODUCTION

Borcherds initiated the Generalized Kac-Moody Algebras (abbreviated as GKM algebras) in [1]. Special and Strictly imaginary roots was introduced by Bennett and Casperson in [2] and [3]. A new class called Super Hyperbolic (abbreviated as SH) type of KM algebra was introduces by Chen Hongji and Liu Bin in [4]. In [5], [6], [7], [8], [9], a complete classification of GKM algebras of purely imaginary roots, strictly imaginary roots, special imaginary roots was given and also derived the root multiplicity formula for some finite, affine and indefinite type of GKM algebras. Uthra and Priyanka Studied the root structure of Super Hyperbolic Kac-Moody algebras and Generalized Kac-Moody algebras for the families $SH_{18}^{(3)}$, $SH_{71}^{(3)}$, $SHGGH_2^{(3)}$ , $SHGGH_{18}^{(3)}$ and $SHGGH_{71}^{(3)}$ in [10]-[14]. Xinfang Song, et.al determined the root Structure and root multiplicity of a special GKM algebras for the family $EB_2$ in [15] and [16].

## II. PRELIMINARIES

Let I = {1,2,…,n} - a finite index set and $A = (a_{ij})_{i,j\in I}$ - a real n x n matrix is said to be a Generalized Generalized Cartan Matrix (abbreviated as GGCM) of A, if it satisfies the following conditions:

i) $a_{ij}$ = 0 or $a_{ij} \le 0$ for all i ε I, ii) $a_{ij} \le 0$ for i≠j, $a_{ij}$ ε Z if $a_{ii}$ =2 ,

iii) $a_{ij}$ = 0 implies $a_{ij}$ = 0.

In GKM algebras the Dynkin diagrams is defined as follows: To every GGCM A is associated a Dynkin diagram S(A) defined as follows : S(A) has n vertices and vertices i and j are connected by max {|aij|,|aji|} number of lines if aij.aji $\le 4$ and there is an arrow pointing towards i if |aij|>1. If aij.aji > 4, i and j are connected by a bold faced edge, equipped with the ordered pair (|aij|,|aji|) of integers. If aii = 2, ith vertex will be denoted by a white circle and if aii = 0, ith vertex will be denoted by a crossed circle. If aii = -k, k > 0, ith vertex will be denoted by a white circle with –k written above the circle within the parenthesis.

We define a GGCM $A = (a_{ij})_{i,j=1}^{n}$ is of Super Hyperbolic type (abbreviated as SH type) if A is not of hyperbolic type and any indecomposable proper principal submatrix of A is of finite, affine or hyperbolic type. [4]

Let α be an imaginary root of g(A) which is symmetrizable GKM algebra. We call α a special imaginary root, if α satisfies the following conditions: i) (α, α) ≠ 0; ii)r α ( $\Delta$ ) = $\Delta$, r α ( $\Delta^{re}$ ) = $\Delta^{re}$, r α ( $\Delta^{im}$ ) = $\Delta^{im}$ ; r α preserves root multiplicities.

A root $\gamma \in \Delta^{im}$ is said to be strictly imaginary if for every $\alpha \in \Delta^{re},\ \alpha+\gamma \ or\ \alpha-\gamma$ is a root. The set of all strictly imaginary roots is denoted by $\Delta^{sim}$.

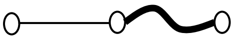
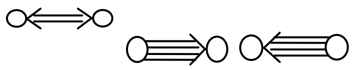
A root $\alpha \in \Delta_+^{im}$ is called purely imaginary if for any $\beta \in \Delta_+^{im},\ \alpha+\beta \in \Delta_+^{im}$ The KM algebra is said to have the purely imaginary property if every imaginary root is purely imaginary.

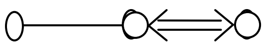
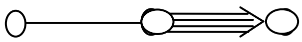
**Note: Throughout this paper, the GGCM consists of one imaginary simple root.**

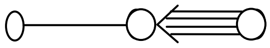
## III. CLASSIFICATION OF DYNKIN DIAGRAMS OF SH GKM ALGEBRAS $SHGGH_n^{(3)}$, n=96,97,98

In this section, the GGCM $\begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ is considered, where $k, d_i, e_i \in \Re_+ \cup \{-2\} \forall i$ and $gh = 4,\ either\ g = 2, h = 2\ or\ g = 1, h = 4\ or\ g = 4, h = 1$

**Proposition 3.1** : There are 1998 non-isomorphic connected Dynkin diagrams for each case (i.e., $SHGGH_{96}^{(3)}$, $SHGGH_{97}^{(3)}$, $SHGGH_{98}^{(3)}$**)** associated with the GGCM of indefinite Super hyperbolic type $SHGGH_n^{(3)}$.

**Proof :** The associated Dynkin diagram with the hyperbolic family $H_n^{(3)}$, where n = 96, 97, 98 is . Here  represents the following three possible edges 

(i.e)., $H_{96}^{(3)}$ :  $H_{97}^{(3)}$ : 

$H_{98}^{(3)}$ : 

By extending the 4th vertex with $H_n^{(3)}$, where n = 96,97,98 and all the possible combinations of connected non-isomorphic Dynkin diagrams are determined for the associated GGCM of SH type $SHGGH_n^{(3)}$. Where n = 96,97,98. Here  is represented by one of the possible 9 edges:

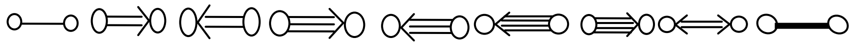
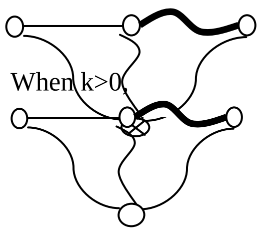
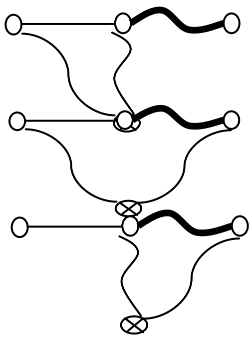
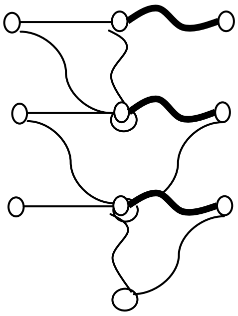
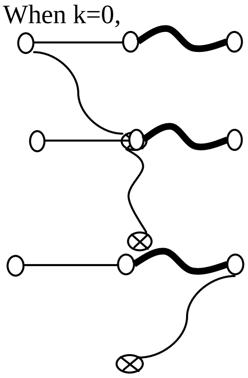
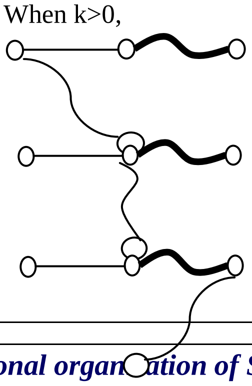


**Table 1**

| Extended Dynkin diagram of Super hyperbolic type $SHGGH_{18}^{(3)}$ | Corresponding GGCM | Number of possible Dynkin diagrams |
|---|---|---|
| When k=0,<br>When k>0,<br> | $\begin{pmatrix} 0 & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$<br>$\begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ | In this case, we extend the 4th vertex to all the other three vertices and we get $9^3$ connected Dynkin diagrams.<br>(i.e), there exists 729 Dynkin diagrams for both the case.<br>Thus, 729 Dynkin diagrams when k=0 and 729 Dynkin diagrams when k > 0. Totally, we get 1458 Dynkin diagrams for both the cases. |

| | | |
|---|---|---|
| When k=0,  | $\begin{pmatrix} 0 & -d_1 & -d_2 & 0 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ 0 & 0 & -h & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & -d_1 & 0 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & -d_2 & -d_3 \\ 0 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ | In this case, among the 3 vertices, the fourth vertex is connected with any of the two vertices by the 9 possible edges with different combinations. Therefore, the associated Dynkin diagrams exists in this case is 2 x $9^2$ = 243 non-isomorphic Dynkin diagrams. (i.e)., 243 Dynkin diagrams, when k=0 and 243 Dynkin diagrams when k>0 Therefore, totally we get, 486 non-isomorphic connected Dynkin diagrams. |
| When k>0,  | $\begin{pmatrix} -k & -d_1 & -d_2 & 0 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ 0 & 0 & -h & 2 \end{pmatrix}$ $\begin{pmatrix} -k & -d_1 & 0 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ $\begin{pmatrix} -k & 0 & -d_2 & -d_3 \\ 0 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ | |
| When k=0,  | $\begin{pmatrix} 0 & -d_1 & 0 & 0 \\ -e_1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -g \\ 0 & 0 & -h & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & -d_2 & 0 \\ 0 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ 0 & 0 & -h & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 & 0 & -d_3 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ | In this case, the fourth vertex is connected independently to the other three vertices by the 9 possible edges. Therefore, the possible number of non-isomorphic connected Dynkin diagrams associated with $SHGGH_n^{(3)}, n = 96,97,98$ is 3 x 9 = 27. (i.e)., 27 Dynkin diagrams, when k=0 and 27 Dynkin diagrams when k>0 Therefore, totally we get, 54 non-isomorphic connected Dynkin diagrams. |
| When k>0,  | $\begin{pmatrix} -k & -d_1 & 0 & 0 \\ -e_1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -g \\ 0 & 0 & -h & 2 \end{pmatrix}$ $\begin{pmatrix} -k & 0 & -d_2 & 0 \\ 0 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ 0 & 0 & -h & 2 \end{pmatrix}$ | |

| | | |
|---|---|---|
| | $\begin{pmatrix} -k & 0 & 0 & -d_3 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ | |

Thus there exists 54 + 486 + 1458 = 1998 types of connected, non isomorphic Dynkin diagrams associated with the GGCM of $SHGGH_n^{(3)}, n = 96,97,98$.

**Note:** In the GGCM of table 1, $gh = 4,\ (i.e).,\ either\ g = 2, h = 2\ or\ g = 1, h = 4\ or\ g = 4, h = 1$.

## IV. IMAGINARY ROOT SYSTEM OF $SHGGH_n^{(3)}$, N=96,97,98

Consider the symmetrizable GGCM of $SHGGH_n^{(3)} = \begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ where n = 96,97,98 and $gh = 4,\ (i.e).,\ either\ g = 2, h = 2\ or\ g = 1, h = 4\ or\ g = 4, h = 1$ with the conditions $d_1e_2 = e_1d_2\ and\ d_2e_3 = d_3e_2$ where $k, d_i, e_i \in \Re_+ \cup \{-2\} \forall i$.

We have $\prod = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, $\prod^{re} = \{\alpha_2, \alpha_3, \alpha_4\}\ and\ \prod^{im} = \{\alpha_1\}$.

The non-degenerate symmetric bilinear form are given as, $(\alpha_1,\alpha_1) = -ke_1, (\alpha_1,\alpha_2) = -d_1e_1, (\alpha_1,\alpha_3) = -d_1e_2, (\alpha_1,\alpha_4) = -d_1e_3, (\alpha_2,\alpha_2) = 2d_1,$ $(\alpha_2,\alpha_3) = -d_1, (\alpha_2,\alpha_4) = 0, (\alpha_3,\alpha_3) = 2d_1, (\alpha_3,\alpha_4) = -gd_1, (\alpha_4,\alpha_4) = 2d_1$

The fundamental reflections are

$r_2(\alpha_1) = \alpha_1 + e_1d_1\alpha_2, r_2(\alpha_2) = -\alpha_2, r_2(\alpha_3) = \alpha_3 + \alpha_2, r_2(\alpha_4) = \alpha_4,$

$r_3(\alpha_1) = \alpha_1 + e_2d_1\alpha_3, r_3(\alpha_2) = \alpha_2 + \alpha_3, r_3(\alpha_3) = -\alpha_3, r_3(\alpha_4) = \alpha_4 + g\alpha_3,$

$r_4(\alpha_1) = \alpha_1 + e_3d_1\alpha_4, r_4(\alpha_2) = \alpha_2, r_4(\alpha_3) = \alpha_3 + h\alpha_4, r_4(\alpha_4) = -\alpha_4$

Here, $\Delta_+^{im} = \bigcup_{w \in W} w(K)$ where K is given by

$K = \{k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 / k_1 \in \mathrm{N}, k_2, k_3, k_4 \in \mathrm{Z}_+,$

$2k_2 \le e_1k_1 + k_3, 2k_3 \le e_2k_1 + k_2 + gk_4\ and\ 2k_4 \le e_3k_1 + hk_3\ with$

$k_2 = 0 \Rightarrow 2k_3 \le e_2k_1 + gk_4\ and\ 2k_4 \le e_3k_1 + hk_3,$

$k_3 = 0 \Rightarrow 2k_2 \le e_1k_1\ and\ 2k_4 \le e_3k_1$

$k_4 = 0 \Rightarrow 2k_2 \le e_1k_1 + k_3\ and\ 2k_3 \le e_2k_1 + k_2,$

$k_2 = k_3 = 0 \Rightarrow 2k_4 \le m_3k_1;$

$k_2 = k_4 = 0 \Rightarrow 2k_3 \le e_2k_1;\ \ k_2 = k_3 = k_4 = 0 \Rightarrow k_1 = 1\}$

Therefore, the Weyl group is infinite.

**Proposition 4.1:** Let A=$\begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ be the symmetrizable GGCM of $SHGGH_n^{(3)}, n = 96,97,98$, $gh = 4,\ (i.e).,\ either\ g = 2, h = 2\ or\ g = 1, h = 4\ or\ g = 4, h = 1$

with the conditions $d_1e_2 = e_1d_2\ and\ d_2e_3 = d_3e_2$ where $k, d_i, e_i \in \Re_+ \cup \{-2\} \forall i$. There exists no special imaginary root of g(A).

**Proof:** Suppose $\alpha = \sum_{i-1}^{n+1} k_i\alpha_i \in K \subseteq \Delta_+^{im}, k_i \in \mathrm{Z}_+ \forall i$ be a special imaginary root of g(A). We have

$$(\alpha,\alpha) = 2d_1k_2^2 + 2d_1k_3^2 + 2d_1k_4^2 - e_1kk_1^2 - 2d_1e_1k_1k_2 - 2d_2e_1k_1k_3$$
$$- 2d_3e_1k_1k_4 - 2d_1k_2k_3 - 2gd_1k_3k_4 \quad < 0$$

Let $(\alpha,\alpha)=A$. By the reflection of imaginary root definition, we have

$$\left.\begin{aligned} r_\alpha(\alpha_1) &= \alpha_1 + \frac{2e_1}{A}(kk_1 + d_1k_2 + d_2k_3 + d_3k_4)\alpha, r_\alpha(\alpha_2) = \alpha_2 - \frac{2d_1}{A}(2k_2 - e_1k_1 - k_3)\alpha \\ r_\alpha(\alpha_3) &= \alpha_3 - \frac{2d_1}{A}(2k_3 - e_2k_1 - k_2 - gk_4)\alpha, r_\alpha(\alpha_4) = \alpha_4 - \frac{2d_1}{A}(2k_4 - e_3k_1 - hk_3)\alpha \end{aligned}\right\}\ldots(1)$$

Then for a special imaginary root α, we have $r_\alpha(\alpha_2) = \alpha_2; r_\alpha(\alpha_3) = \alpha_3; r_\alpha(\alpha_4) = \alpha_4$ … (2)

From the above equations (1) and (2), we get

$$(2k_2 - e_1k_1 - k_3)\alpha = (2k_3 - e_2k_1 - k_2 - gk_4)\alpha = (2k_4 - e_3k_1 - hk_3)\alpha = 0$$

Since $A = \begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -g \\ -e_3 & 0 & -h & 2 \end{pmatrix}$ be the symmetrizable GGCM of $SHGGH_n^{(3)}$, n=96,97,98 and $gh = 4,\ (i.e).,\ either\ g = 2, h = 2\ or\ g = 1, h = 4\ or\ g = 4, h = 1$

Therefore, by using the above results, the following three cases are obtained.

**Case i):** when g=2 and h=2

The symmetrizable GGCM of $SHGGH_{96}^{(3)}$ is given by $A = \begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -2 \\ -e_3 & 0 & -2 & 2 \end{pmatrix}$. Thus we get,

$k_2 = k_1(e_2 + e_3)$ is absurd. Therefore, no special imaginary root exists for $SHGGH_{96}^{(3)}$.

**Case ii):** when g=1 and h=4

The symmetrizable GGCM of $SHGGH_{97}^{(3)}$ is given by $A = \begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -1 \\ -e_3 & 0 & -4 & 2 \end{pmatrix}$. Thus we get,

$k_3 = k_1(-e_1 + 2e_2 + e_3)$ is absurd and no special imaginary root exists for $SHGGH_{97}^{(3)}$.

**Case iii):** when g=4 and h=1

The symmetrizable GGCM of $SHGGH_{98}^{(3)}$ is given by $A = \begin{pmatrix} -k & -d_1 & -d_2 & -d_3 \\ -e_1 & 2 & -1 & 0 \\ -e_2 & -1 & 2 & -4 \\ -e_3 & 0 & -1 & 2 \end{pmatrix}$. Thus we get,

$k_1(4e_3 - 2e_2 - e_1) = 0$ is absurd and there exists no special imaginary root exists for $SHGGH_{98}^{(3)}$.

**Proposition 4.2:** The Super hyperbolic GKM algebra $SHGGH_n^{(3)}$, n=96,97,98 satisfies the purely imaginary property.

**Proof:** In GKM algebras some family possess purely imaginary property and some does not possess. Here the, SH GKM algebra $SHGGH_n^{(3)}$, n=96,97,98 proves the purely imaginary property.

In this family, we have taken $\alpha_1$ to be an imaginary simple root and if any root added with $\alpha_1$, is also an imaginary root and also support of α is connected. Therefore, all imaginary roots are purely imaginary and hence, for any $\alpha \in \Delta_+^{im}$ and for any $\beta \in \Delta_+^{im}$ we get $\alpha + \beta \in \Delta_+^{im}$ is a root.

**Proposition 4.3:** The Super hyperbolic GKM algebra $SHGGH_n^{(3)}$, n=96,97,98 satisfies the strictly imaginary property.

**Proof:** Since support of α is connected, the addition or subtraction of any combination of $\alpha_i$ (i=1,2,3&4) is a root. Hence, for any $\alpha \in \Delta^{re}$ and for any $\gamma \in \Delta_{+}^{im}$ we get $\alpha + \gamma$ is a root. Therefore, the SH GKM algebra $SHGGH_n^{(3)}$, n=96,97,98 satisfies the strictly imaginary property.

## V. CONCLUSION

The work done in this paper, helps us to understand the root system of $SHGGH_n^{(3)}$, n=96,97,98. Similarly, we can further develop this imaginary root system for GKM Super algebras and also root multiplicities can be determined.

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