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Adomian Decomposition Method for Solving Highly Nonlinear Fractional Partial Differential Equations

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Abstract: The aim of this paper is to propose the Adomian Decomposition Method (ADM) to solve linear and nonlinear fractional partial differential equations. The fractional derivatives are described in the Caputo sense. The approximate solution of the partial differential equations shows that the fractional iteration method is very efficient technique. As an application of this method we solve some test problems and their solutions are represented graphically by very powerful software Mathematica

Keywords: Time Fractional Differential Equation, Caputo Fractional Derivative, Adomian Decomposition Method, Mathematica

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I. I.INTRODUCTION

Fractional calculus has emerged as one of the most important interdisciplinary subjects in Mathematics, Physics, Biology and Engineering. The theory of derivatives of non-integer order was discovered by Leibniz in 1695 [10] Leibniz's note led to the theory of fractional calculus, which was developed by Liouville, Grunwald, Letnikov and Riemann in 19th century [11, 12, 13, 14, 15].

For three centuries the theory of fractional derivatives developed as a purely theoretical field of Mathematics, useful only for mathematicians. However, in the last few decades many authors pointed out that derivatives and integrals of noninteger order are very suitable for the description of properties of real materials e.g. polymers etc. It has shown that new fractional order models are more adequate than previously used integer order models [5, 20].

Nonlinear Partial Differential Equations have remarkable developments in different areas like gravitation, chemical reaction, fluid dynamics, dispersion, nonlinear optics, plasma physics, acoustics etc. Nonlinear wave propagation problems have provided solutions of different physical structures than solutions of linear wave equations. Nonlinear Partial Differential equations have been widely studied throughout recent years. The importance of obtaining the exact solutions of nonlinear equations in mathematics is still a significant problem that needs new research work. Motivated by the advancement of fractional calculus, many researchers have focused to investigate the solutions of nonlinear differential equations with the fractional operator by developing quite a few analytical or numerical techniques to find approximate solutions [4, 6, 9, 16, 17, 18]. These differential equations involve several fractional differential operators like Riemann-Liouville, Caputo, Hilfer etc. [3, 8, 19].

Over the last 25 years the Adomian Decomposition Method [1, 2] has been applied to obtain a formal solution to a wide class of both deterministic and stochastic Partial Differential Equations. In recent years, this Method has emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, integral, integro-differential, higher order ordinary differential equations, partial differential equations. The main advantage of the method is that it can provide analytical or an approximated solution to a wide class of nonlinear equations without linearization, perturbation or discretization methods. Another important advantage is that the method is capable of greatly reducing the size of computation work, while still maintaining high accuracy of the numerical solution.

We organize this paper as follows: In section 2, we define some basic preliminaries and properties of fractional calculus. Section 3, is developed for detailed analysis of fractional Adomian Decomposition Method. In the section 4, we present some examples to show the applicability and efficiency of the method and also their solutions are demonstrated with the help of software Mathematica. In the last section conclusion is given.

II. PRELIMINARIES AND NOTATIONS

In this section, we study some definitions and properties of fractional calculus.

Definition 2.1 *A real function $f(t)$, $t > 0$, is said to be in the space C_α, $\alpha \in R$ if there exists a real number $p > \alpha$, such that $f(t) = t^p f_1(t)$, where $f_1(x) \in C[0,\infty)$ and it is said to be in the space C_α^m if and only if $f^{(m)}(t) \in C_\alpha$, $m \in N$.*

Definition 2.2 *The Caputo derivative of fractional order α of a function $f(t)$, $f(t) \in C_{-1}^m$ is defined as follows*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(\tau)}{(t-\tau)^{(1-m+\alpha)}} d\tau, \qquad \textit{for} \quad m-1 < \alpha \leq m,\ m \in N, x > 0,$$

(i) $D_x^\beta sinx = sin(x + \beta\frac{\pi}{2})$,

(ii) $D_x^\beta cosx = cos(x + \beta\frac{\pi}{2})$.

Definition 2.3 *The Riemann - Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \ \alpha > 0, \ x > 0$$

$$J^0 f(x) = f(x).$$

Properties:

For $f(x) \in C_\mu$, $\mu \geq -1$, $\alpha,\beta \geq 0$ and $\gamma > -1$, we have

(i) $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$,

(ii) $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$.

Properties:

It is simple to prove the following properties of fractional derivatives and integrals that will be used in the analysis

(i) $D^\alpha J^\alpha f(t) = f(t)$,

(ii) $J^\alpha D_*^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{t^k}{k!}, x > 0$,

(iii) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{(\gamma+\alpha)}$.

(iv) $D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{(\gamma-\alpha)}$

In the next section, we develop the fractional Adomain decomposition method for fractional partial differential equation.

III. THE FRACTIONAL ADOMIAN DECOMPOSITION METHOD (FADM)

In order to elucidate the solution procedure of the FADM, we consider the following general fractional partial differential equation

$$L^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t),\ m-1 < \alpha \leq m,\ x > 0,\ t > 0 \qquad (3.1)$$

where L is fractional order derivative, R is linear differential operator, N is nonlinear operator and g is source term. Let

$$L^\alpha = \frac{\partial^{n\alpha}}{\partial t^{n\alpha}} \qquad (3.2)$$

be the $(n\alpha)^{th}$ order fractional derivative then the corresponding $L^{-\alpha}$ operator will be written in the following form

$$J^\alpha = L^{-\alpha} = \frac{1}{\Gamma^n(\alpha+1)} \int_0^t \int_0^{\tau_n} \int_0^{\tau_{(n-1)}} \ldots\ldots\ldots \int_0^{\tau_2} (d\tau_1)^\alpha (d\tau_2)^\alpha (d\tau_3)^\alpha \ldots\ldots (d\tau_n)^\alpha \quad (3.3)$$

and

$$\frac{1}{\Gamma(\alpha+1)} \int_0^t (d\tau_n)^\alpha$$

is the Caputo integration.

Operating with the operator J^α on both sides of equation (3.1) .we have

$$J^\alpha[L^\alpha u(x,t) + Ru(x,t) + Nu(x,t)] = J^\alpha g(x,t)$$

$$J^\alpha L^\alpha u(x,t) = -J^\alpha[Ru(x,t) - Nu(x,t)] + J^\alpha g(x,t) \qquad (3.4)$$

Using properties in equation (3.4), we get

$$u(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} - J^\alpha[Ru(x,t) + Nu(x,t)] + J^\alpha g(x,t), \ m-1 < \alpha \leq m. \quad (3.5)$$

Now, we decompose the unknown function $u(x,t)$ into sum of an infinite number of components given by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (3.6)$$

The nonlinear terms $Nu(x,t)$ are decomposed in the following form:

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n \quad (3.7)$$

where the Adomian polynomial can be determined as follows:

$$A_n = \frac{1}{n!}\left[\frac{d^n N}{d\lambda^n}(\sum_{k=0}^{n} \lambda^k u_k)\right]_{\lambda=0} \quad (3.8)$$

where A_n is called Adomian polynomial and that can be easily calculated by Mathematica software.
Substituting the decomposition series (3.6) and (3.7) into both sides of equation (3.5) gives

$$\sum_{n=0}^{\infty} u_n(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} + J^\alpha g(x,t) - J^\alpha \left[R \sum_{n=0}^{\infty} u_n(x,t) + \sum_{n=0}^{\infty} A_n\right], \ x > 0 \quad (3.9)$$

The components $u_n(x,t)$, $n \geq 0$ of the solution $u(x,t)$ can be recursively determined by using the relation as follows:

$$u_0(x,t) = \sum_{k=0}^{m-1} \frac{\partial^k u(x,0)}{\partial t^k} \frac{t^k}{k!} + J^\alpha g(x,t), \quad (3.10)$$

$u_1(x,t) = -J^\alpha(Ru_0 + A_0)$
$u_2(x,t) = -J^\alpha(Ru_1 + A_1)$
$u_3(x,t) = -J^\alpha(Ru_2 + A_2)$

…

$u_{n+1}(x,t) = -J^\alpha(Ru_n + A_n)$

where each component can be determined by using the preceding components and we can obtain the solution in a series form by calculating the components $u_n(x,t)$, $n \geq 0$. Finally, we approximate the solution $u(x,t)$ by the truncated series.

$$\phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t)$$

$$\lim_{N\to\infty} \phi_N = u(x,t)$$

In the next section, we illustrate example and its solution is represented graphically by mathematica software.

IV. APPLICATIONS

In order to elucidate the solution procedure of the Adomian decomposition method, we consider following nonlinear time fractional partial differential equations.

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u^3(x,t)}{\partial x^2} \quad (4.1)$$

Initial condition: $u(x, 0) = sinx$
Boundary conditions: $u(0, t) = 0$, $u(1, t) = 0$, $t \geq 0$

In the operator form the time fractional nonlinear equation becomes

$$L_t^\alpha u(x,t) = L_{xx} u^3$$

Initial condition: $u(x, 0) = sinx$
Boundary conditions: $u(0, t) = 0$, $u(1, t) = 0$, $t \geq 0$

By using ADM, we have following recursive relation

$$u_0(x,t) = sinx,$$

$$u_{k+1}(x,t) = J^{\alpha}\left[L_{xx}A_k\right]$$

That leads to

$$u_0(x,t) = sinx$$

Calculating the values of u_1, as follow-

$$u_1(x,t) = J^{\alpha}\left[L_{xx}A_0\right]$$

$$A_0 = u_0^3,$$

$$A_0 = (sinx)^3,$$

$$u_1(x,t) = J^{\alpha}\left[L_{xx}(sinx)^3\right],$$

$$u_1(x,t) = \left[\frac{-3}{4}sinx + \frac{9}{4}sin3x\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

Calculating the values of u_2, as follow-

$$u_2(x,t) = J^{\alpha}\left[L_{xx}A_1\right]$$

$$A_1 = 3u_0^2u_1,$$

$$A_1 = \left[\frac{-9}{4}sin^3x + \frac{27}{4}sin^2xsin3x\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = J^{\alpha}\left[L_{xx}\left(\frac{-9}{4}sin^3x + \frac{27}{4}sin^2xsin3x\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right],$$

$$u_2(x,t) = \left[\frac{-567}{16}sin3x + \frac{675}{16}sin5x\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

...

Therefore, the series solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...$$

Substituting values of components in above equation, we get the solution of Nonlinear Fractional Partial Differential Equation.

$$u(x,t) = sinx + \left[\frac{-3}{4}sinx + \frac{9}{4}sin3x\right]\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \left[\frac{-567}{16}sin3x + \frac{675}{16}sin5x\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots$$

The graphical representation of solution by Mathematica software:

Figs 1 and 2 shows the numerical simulations of three term approximate solutions of (4.1) for $\gamma = 1$, 0.8 using Adomian decomposition method which infers that this technique can foresee the conduct of said variables precisely for the considered region. The intervals [0, 1.5] and [0, 1] gives the valid region of convergence of solutions for $\gamma = 1$ and intervals [0, 2], [0, 2.1] for $\gamma = 0.8$.

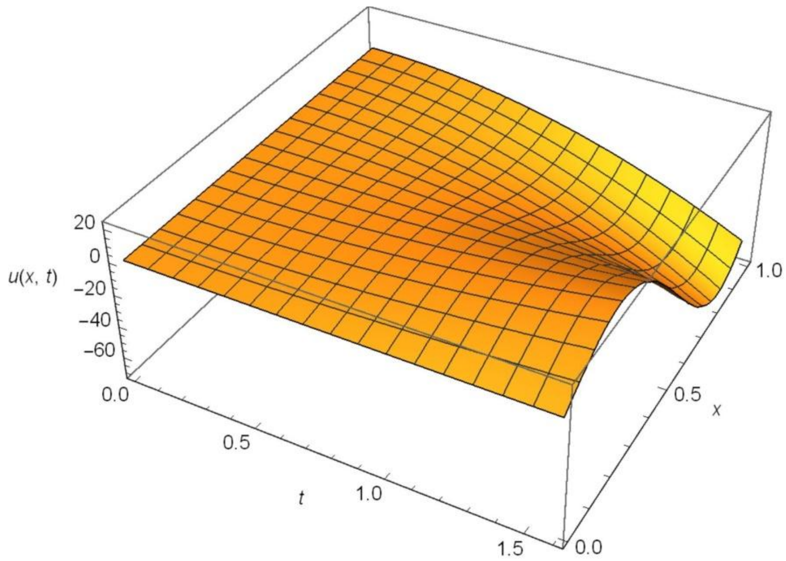

Figure 1: Approx. sol. of (4.1) for $\alpha = 1$

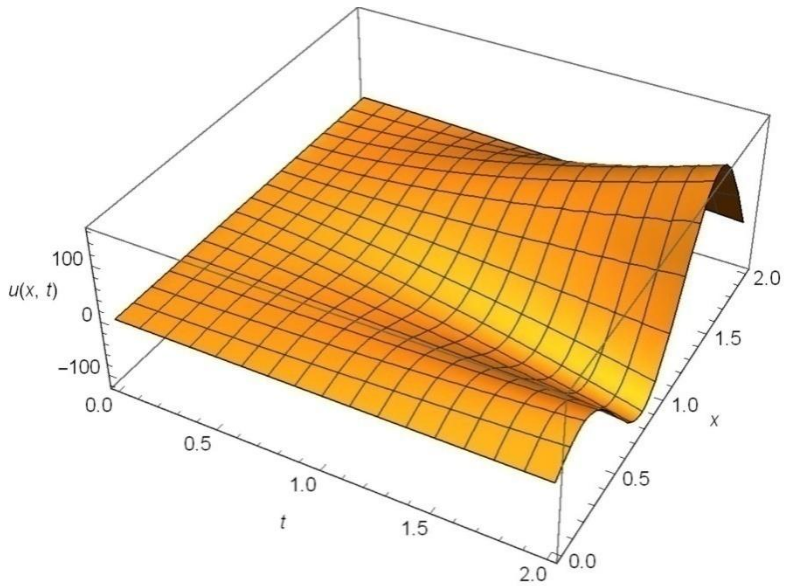

Figure 2: Approx. sol. of (4.1) for $\alpha = 0.8$

V. CONCLUSION

The main objective of this work is to obtain a solution for nonlinear fractional partial differential equation. We observe that Adomian Decomposition Method is a powerful method to solve nonlinear fractional partial differential equation. To, show the applicability and efficiency of the proposed method, the method is applied to obtain the solutions of several examples. The obtained results demonstrate the reliability of the algorithm. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods. Finally, we come to the conclusion that the Adomian Decomposition Method is very powerful and efficient in finding solutions for wide class of nonlinear fractional partial differential equation.

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