IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 09, Issue 5 (May. 2019), ||S (III) || PP 19-26

Role of Alternative Prey in A Predator-Prey Model With Prey Refugia - A Mathematical Study

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Abstract: Here, we have studied mathematically the role of alternative prey in a predator-prey model with prey refugia. We have discussed the local and global stability of our system around different equilibrium points. Some numerical simulations are presented to shown significant qualitative results. Our results shown that alternative prey for predator population has an important role to control the system dynamics.

Keywords: Alternative prey, refugia, stability , stable focus, limit cycle, bifurcation.

Date of Submission: 17-05-2019 Date of acceptance: 03-06-2019

I. INTRODUCTION

After the pioneering work of Lotka[2] and Volterra[18], the study of predator-prey model becomes much familiar. Although the study of ecological model has seen much progress in last few decades, many long standing mathematical and ecological problems remain open. The mathematical models of predator population and prey population depend on various factors and parameters. Many researchers[3, 4, 9, 10, 11, 16] study predator-prey ecological model introducing various techniques such as disease, refugia, harvesting, grouping, allee effects etc. Chattopadhyay and Arino [7] studied a predator-prey model introducing disease in the prey species. Hasting and Powell[1] studied the chaotic nature of a three species food chain model. Biswas et al[14, 15] studied the dynamical behaviour of an eco-epidemiolocal model with allee effects and harvesting. Das et al[8] discuss a predator-prey model where both populations are affected by disease. Roy et al[12] studied the chaotic behaviour of a predator-prey food chain model with disease in intermediate population.

In nature, all living species like a suitable environment where it can live freely and reproduce. Ecological species take various techniques for searching foods and for defensive purposes. Predator population depends on their prey for survival and prey population applies different refugia strategies for living. Wang et al[6] proved that the prey species are able to refuge their predators using some techniques. Kar[17] discuss the activities of harvesting and prey refuge by studing a prey-predator model. Chen et al[5] showed that the densities of prey and predator populations can be affected by introducing prey refuge in a system. For this reason, when prey refuse is very high predator has no food source and go for extinction. Now, if the predator population has an alternative prey sources for food, they can survive even prey refuse is high. Alternative prey creates a positive impact for sustaining predator population. From this point of view, we have tried to study the effects of alternative prey in an ecological predator-prey system with prey refuse. The main objective of this study is to investigate mathematically the dynamic properties and behaviours of the system.

The paper is organized as follows. In the section(2), we outline the mathematical model with some basic assumptions. We study the local and global stability of the system about different equilibrium points in section(3). We perform some numerical simulations and discussion in section (4). The article ends with a conclusion.

II. MODEL FORMULATION

Here we consider a predator-prey food chain model with prey refuse and alternative prey for predator population. Let x(t) denotes the prey population, y(t) denotes alternative prey and z(t) denotes the predator population. We take the following assumption to formulate our model:

(a) When the predator population is absent, the prey population grows logistically with intrinsic growth rate ‘p’ and carrying capacity’ k’(> 0).
(b) The growth rate of alternative prey is ‘q’.
(c) We take the interaction functions between (i) predator and prey, (ii) alternative prey and predator are of Holling type-II.

With the above assumptions we write our mathematical model as the following set of nonlinear ordinary differential equations:

$$\begin{aligned}
\frac{dx}{dt} &= px\left(1 - \frac{x}{k}\right) - \frac{a_1(1-m)xz}{1+b_1(1-m)x} \\
\frac{dy}{dt} &= qy(1-y) - \frac{a_2yz}{1+b_2y} \\
\frac{dz}{dt} &= \frac{e_1 a_1(1-m)xz}{1+b_1(1-m)x} + \frac{e_2 a_2yz}{1+b_2y} - dz
\end{aligned} \tag{1}$$

Here, the constants a_1 and a_2 are the maximal predation rates of predator population for prey and alternative prey respectively; e_1 and e_2 are conversion rates of prey and alternative prey to predator respectively; b_1 and b_2 are half saturation constants for prey and alternative prey respectively; m is the refugia parameter for prey population; d is the date rate of predator population.

System (1) has to be analyzed with the following initial conditions:
$x(0) > 0$; $y(0) > 0$; $z(0) > 0$.

III. MODEL ANALYSIS

3.1. Existence of equilibrium points

The system has seven equilibrium points. The trivial equilibrium point $E_0(0, 0, 0)$, the axial equilibrium points $E_1(k, 0, 0)$ and $E_2(0, 1, 0)$, the predator free equilibrium point is $E_3(k, 1, 0)$ exist for all parametric values. Alternative prey free equilibrium point is $E_4(x_4, 0, z_4)$ where,

$x_4 = \dfrac{d}{(1-m)(e_1a_1 - b_1d)}$ and $z_4 = \dfrac{px_4\left(1 - \frac{x_4}{k}\right)(1 + b_1(1-m)x_4)}{a_1(1-m)}$. The existence condition of E_4 is $e_1a_1 - b_1d > 0$ since $(1 - m) > 0$.

The prey species free equilibrium point is $E_5(0, y_5, z_5)$ where,

$y_5 = \dfrac{d}{(e_2 a_2 - b_2d)}$ and $z_5 = \dfrac{q}{a_2}(1 - y_5)(1 + b_2 y_5)$. The existence condition of E_5 is $e_2 a_2 - b_2d > 0$.

The interior equilibrium point is given by $E^*(x^*, y^*, z^*)$ where x^*, y^* and z^* satisfies the following set of equations:

$$p\left(1 - \frac{x^*}{k}\right) - \frac{a_1(1-m)z^*}{1+b_1(1-m)x^*} = 0$$

$$q(1 - y^*) - \frac{a_2z^*}{1+b_2 y^*} = 0$$

$$\frac{e_1 a_1(1-m)x^*}{1+b_1(1-m)x^*} + \frac{e_2 a_2y^*}{1+b_2y^*} - d = 0$$

3.2. Local stability around the boundary equilibrium points

The Jacobian matrix J of the system (1) at any arbitrary point (x, y, z) is given by $(J_{ij})_{3\times3}$ where,

$J_{11} = p\left(1 - \dfrac{2x}{k}\right) - \dfrac{a_1(1-m)z}{(1+b_1(1-m)x)^2}$, $J_{12} = 0$, $J_{13} = -\dfrac{a_1(1-m)x}{1+b_1(1-m)x}$, $J_{21} = 0$, $J_{22} = q(1-2y) - \dfrac{a_2z}{(1+b_2 y)^2}$,

$J_{23} = \dfrac{-a_2y}{1+b_2 y}$, $J_{31} = \dfrac{e_1a_1(1-m)z}{(1+b_1(1-m)x)^2}$, $J_{32} = \dfrac{e_2a_2z}{(1+b_2 y)^2}$, $J_{33} = \dfrac{e_1a_1(1-m)x}{1+b_1(1-m)x} + \dfrac{e_2a_2y}{1+b_2 y} - d$.

Theorem 1. The trivial equilibrium point $E_0(0, 0, 0)$, the axial equilibrium points $E_1(k, 0, 0)$ and $E_2(0, 1, 0)$ are always unstable.

Proof. The eigen values associated with the Jacobian matrix computed around E_0 are p, q and - d. Since both the eigen values $p > 0$ and $q > 0$ so the equilibrium point E_0 is always unstable.

Again, the characteristic roots of the Jacobian matrix computed around E_1 are $-p$, $\dfrac{e_1a_1(1-m)k}{1 + b_1(1-m)k} - d$ and q.

Since one of the eigen value $q > 0$, then the equilibrium point E_1 is always unstable.
The eigen values associated with the Jacobian matrix computed around E_2 are p, $-q$ and $\frac{e_2a_2}{1+b_2} - d$. Since one eigen value $p > 0$, then the equilibrium point E_2 is always unstable.

Theorem 2. The predator free equilibrium point $E_3(k, 1, 0)$ is locally asymptotically stable if $R_{03} < 1$, where $R_{03} = \frac{e_1a_1(1 - m)k(1 + b_2)}{(1 + b_2(1 - m)k)(d + db_2 - e_2a_2)}$. The alternative prey free equilibrium point $E_4(x_4, 0, z_4)$ is locally asymptotically stable if $R_{04} < 1$, where $R_{04} = \frac{q}{a_2z_4}$ with $H_{11} < 0$ and $H_{33} < 0$. The prey species free equilibrium point $E_5(0, y_5, z_5)$ is locally asymptotically stable if $R_{05} < 1$, where $R_{05} = \frac{p}{(1 - m)a_1z_5}$ with $B_{22} < 0$ and $B_{33} < 0$.

Proof. The characteristic roots of the Jacobian matrix computed around the predator free equilibrium point $E_3(k, 1, 0)$ are $-p$, $-q$ and $\frac{e_1a_1(1 - m)k}{1 + b_2(1 - m)k} + \frac{e_2a_2}{1+b_2} - d$. Hence the equilibrium point E_3 will be locally asymptotically stable if $R_{03} < 1$, where $R_{03} = \frac{e_1a_1(1 - m)k(1 + b_2)}{(1 + b_2(1 - m)k)(d + db_2 - e_2a_2)}$.

Let, the Jacobian matrix computed around the alternative prey free equilibrium point $E_4(x_4, 0, z_4)$ is $J_4 = (H_{ij})_{3\times3}$ where, $H_{11} = p(1 - \frac{2x_4}{k}) - \frac{a_1(1 - m)z_4}{(1 + b_1(1 - m)x_4)^2}$, $H_{12} = 0$, $H_{13} = -\frac{a_1(1 - m)x_4}{1 + b_1(1 - m)x_4}$, $H_{21} = 0$, $H_{22} = q - a_2z_4$, $H_{23} = 0$, $H_{31} = \frac{e_1a_1(1 - m)z_4}{(1 + b_1(1 - m)x_4)^2}$, $H_{32} = e_2a_2z_4$, $H_{33} = \frac{e_1a_1(1 - m)z_4}{1 + b_1(1 - m)x_4} - d$.
Now, the characteristic roots of the Jacobian matrix J_4 computed around the equilibrium point E_4 are $q - a_2z_4$ and the roots of the equation
$\eta^2 - (H_{11} + H_{33})\eta + (H_{11} H_{33} - H_{13} H_{31}) = 0$.
Hence, all the characteristic roots of the Jacobian matrix J_4 will be negative if $R_{04} < 1$, where $R_{04} = \frac{q}{a_2z_4}$ with $H_{11} < 0$ and $H_{33} < 0$. Therefore, the system will be locally asymptotically stable around E_4 if the conditions stated above are satisfied.

Again, let, the Jacobian matrix computed around the prey species free equilibrium point E_5 is $J_5 = (B_{ij})_{3\times3}$ where, $B_{11} = p - a_1(1 - m)z_5$, $B_{12} = 0$, $B_{13} = 0$, $B_{21} = 0$, $B_{22} = q(1 - 2y_5) - \frac{a_2z_5}{(1+b_2 y_5)^2}$, $B_{23} = -\frac{a_2z_5}{1+b_2 y_5}$,
$B_{31} = e_1a_1(1 - m)z_5$, $B_{32} = \frac{e_2a_2z_5}{(1+b_2 y_5)^2}$, $B_{33} = \frac{e_2a_2z_5}{1+b_2 y_5} - d$.
The characteristic roots of the Jacobian matrix J_5 is $p - a_1(1 - m)z_5$ and the roots of the equation
$\theta^2 - (B_{22} + B_{33})\theta + B_{22}B_{33} - B_{23}B_{32} = 0$.
Therefore, all the eigen values of the Jacobian matrix J_5 will be negative if $R_{05} < 1$, where $R_{05} = \frac{p}{(1 - m)a_1z_5}$ with $B_{22} < 0$ and $B_{33} < 0$.
Hence, it is obvious that the system will be locally asymptotically stable around the prey species free equilibrium point E_5 if the conditions stated above are satisfied.

3.3. Local stability around the interior equilibrium point $E^*(x^*, y^*, z^*)$

Let, the Jacobian matrix calculated around the interior equilibrium point $E^*(x^*; y^*; z^*)$ is $J^* = (A_{ij})_{3\times3}$, where
$A_{11} = p(1 - \frac{2x^*}{k}) - \frac{a_1(1 - m)z^*}{(1 + b_1(1 - m)x^*)^2}$, $A_{12} = 0$, $A_{13} = -\frac{a_1(1 - m)x^*}{1 + b_1 (1 - m)x^*}$, $A_{21} = 0$, $A_{22} = q(1 - 2y^*) - \frac{a_2z^*}{(1+b_2 y^*)^2}$,
$A_{23} = \frac{- a_2y^*}{1+b_2 y^*}$, $A_{31} = \frac{e_1a_1(1 - m)z^*}{(1+b_1(1 - m)x^*)^2}$, $A_{32} = \frac{e_2a_2z^*}{(1+b_2 y^*)^2}$, $A_{33} = 0$.

Now, the characteristic equation of the matrix J^* is given by $\lambda^3 + \Omega_1\lambda^2 + \Omega_2\lambda + \Omega_3 = 0$, Where, $\Omega_1 = -(A_{11} + A_{22})$

$\Omega_2 = A_{11}A_{22} - A_{13}A_{31} - A_{23}A_{32}$

$\Omega_3 = A_{22}A_{13}A_{31} + A_{11}A_{23}A_{32}$

$\Omega_1\Omega_2 - \Omega_3 = -(A_{11} + A_{22})\, A_{11}A_{22} + A_{11}A_{13}A_{31} + A_{22}A_{23}A_{32}$

Therefore, the interior equilibrium point E* will be asymptotically stable if Ω_1, Ω_2, Ω_3 satisfy all the Routh-Hurwitz conditions (i) all $\Omega_1, \Omega_2, \Omega_3 > 0$ and (ii) $\Omega_1\Omega_2 > \Omega_3$.
Now, all the Routh-Hurwitz conditions will be satisfied if $A_{11} < 0$ and $A_{22} < 0$, i.e. if
$\alpha^2 p(k - 2x^*) < ka_1(1 - m)z^*$ and $\beta^2 q(1 - 2y^*) < a_2 z^*$ where $\alpha = 1+b_1(1-m)x^*$ and $\beta = 1+b_2 y^*$.

3.4. Analysis of global stability around interior equilibrium point E*(x*, y*, z*)

Theorem 3: The interior equilibrium point E*(x*, y*, z*) with respect to system(1) will be globally stable(asymptotically) if the following conditions hold

(i) $z^* < \min\left[\dfrac{pD}{ka_1b_1(1-m)^2}, \dfrac{qD_1}{a_2b_2}\right]$

(ii) $e_1 < 1+b_1(1-m)x^*$

(iii) $e_2 < 1+b_2 y^*$.

Proof. To study the global stability nature of the system(1) around the interior equilibrium point E*(x*, y*, z*) we consider the following positive definite Lyapunov function

$$L(x, y, z) = x + x^* \log\left(\frac{x^*}{x}\right) + y + y^* \log\left(\frac{y^*}{y}\right) + z + z^* \log\left(\frac{z^*}{z}\right).$$

Now, calculating time derivative of L(x, y, z), we get

$$\frac{dL}{dt} = (x - x^*)\left[p\left(1 - \frac{x}{k}\right) - \frac{a_1(1-m)z}{1+b_1(1-m)x}\right] + (y - y^*)\left[q(1 - y) - \frac{a_2 z}{1+b_2 y}\right] + (z - z^*)\left[\frac{e_1a_1(1-m)x}{1+b_1(1-m)x} + \frac{e_2a_2y}{1+b_2y} - d\right]$$

$$= (x - x^*)\left[p\left(\frac{x^*}{k} - \frac{x}{k}\right) + \frac{a_1(1-m)z^*}{1+b_1(1-m)x^*} - \frac{a_1(1-m)z}{1+b_1(1-m)x}\right] + (y - y^*)\left[q(y^* - y) + \frac{a_2z^*}{1+b_2y^*} - \frac{a_2z}{1+b_2y}\right] +$$
$$(z - z^*)\left[\frac{e_1a_1(1-m)x}{1+b_1(1-m)x} - \frac{e_1a_1(1-m)x^*}{1+b_1(1-m)x^*} + \frac{e_2a_2y}{1+b_2y} - \frac{e_2a_2y^*}{1+b_2y^*}\right].$$

After some algebraic calculations, we obtain

$$\frac{dL}{dt} = -\left[\frac{p}{k} - \frac{b_1a_1(1-m)^2z^*}{D}\right](x - x^*)^2 - \left[\frac{b_1a_1(1-m)^2x^* + a_1(1-e_1)(1-m)}{D}\right](x - x^*)(z - z^*) - \left[q - \frac{b_2a_2z^*}{D_1}\right](y - y^*)^2$$
$$- \left[\frac{a_2 + b_2a_2y^* - e_2a_2}{D_1}\right](y - y^*)(z - z^*),$$ where $D = (1+b_1(1-m)x)(1+b_1(1-m)x^*)$ and $D_1 = (1+b_2y)(1+b_2y^*)$.

Therefore, if the conditions(stated above) of the theorem hold, the above expression will be negative definite and the function L(x, y, z) will be a Lyapunov function around the interior equilibrium point E*(x*, y*, z*) which is positive definite.
Hence, we can conclude that, the system(1) with respect to interior equilibrium point E*(x*, y*, z*) will be globally asymptotically stable if the conditions of the theorem hold.

IV. NUMERICAL RESULTS AND DISCUSSION

In this study we perform some numerical experiments to observe the dynamical behavior of the model system using MATLAB software. In this study, alternative prey for predator population is new modification. We have taken a set of hypothetical parameter values $a_1 = 0.6$, $b_1 = 0.02$, $b_2 = 0.5$, $m = 0.32$, $e_1 = 0.02$, $e_2 = 0.4a_2$, $p = 10.0$, $q = 15.0$, $k = 100.0$ and $d = 0.09$. Throughout this numerical experiment we have fixed the above set of hypothetical parameter values. We mainly want to observe the role of alternative prey of the model system.

We observe periodic oscillations of the species for above set of parameter values with $a_2 = 0.33$(Figure(1)). Now we observe trajectory of periodic oscillations in phase plane and we observe limit cycle(Figure(2)). From Figure(3) it is found that the system enters into stable oscillations from periodic oscillations for $a_2 = 0.56$ and in phase plane we found stable focus Figure(4). Therefore if we increase the alternative prey consumption of predator population then the system enters into a stable state from periodic oscillations. To observe clear dynamics for variation of maximal predation rate for alternative prey we draw bifurcation diagram(Figure(5)) and from this diagram we observe that when maximal predation rate is low the system shows periodic oscillations and when maximal predation rate is high the system shows stable focus.

Hence it is clear that the maximal predation rate for alternative prey has a great impact and alternative prey for predator population can stable a food chain model.

V. CONCLUSION

Recently, many researchers study predator-prey model with various biological factors and conditions. In the present paper we consider a predator-prey food chain model with alternative prey source for predator population. We study the local as well as global stability of the model system around different equilibrium points. To study the global dynamics of the model system we perform some numerical simulations and we observe the dynamical behavior of our model system for variation of maximal predation rate for alternative prey. From our study it is clear that alternative prey source for predator population can give us a stable predator-prey food chain model. Alternative prey can prevent the predator population from extinction. Hence our study can open a new window of study in future.

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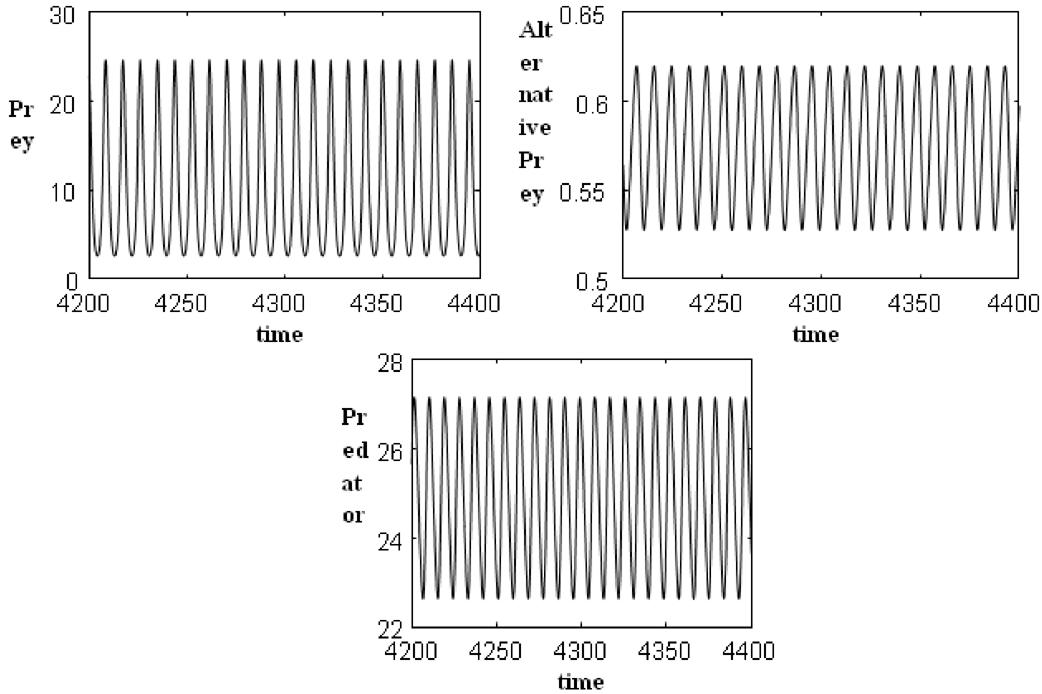

Figure 1: The figure shows the periodic solution of the system (1) for $a_1 = 0.6$, $a_2 = 0.33$, $b_1 = 0.02$, $b_2 = 0.5$, $m = 0.32$, $e_1 = 0.02$, $e_2 = 0.4a_2$, $p = 10.0$, $q = 15.0$, $k = 100.0$ and $d = 0.09$.

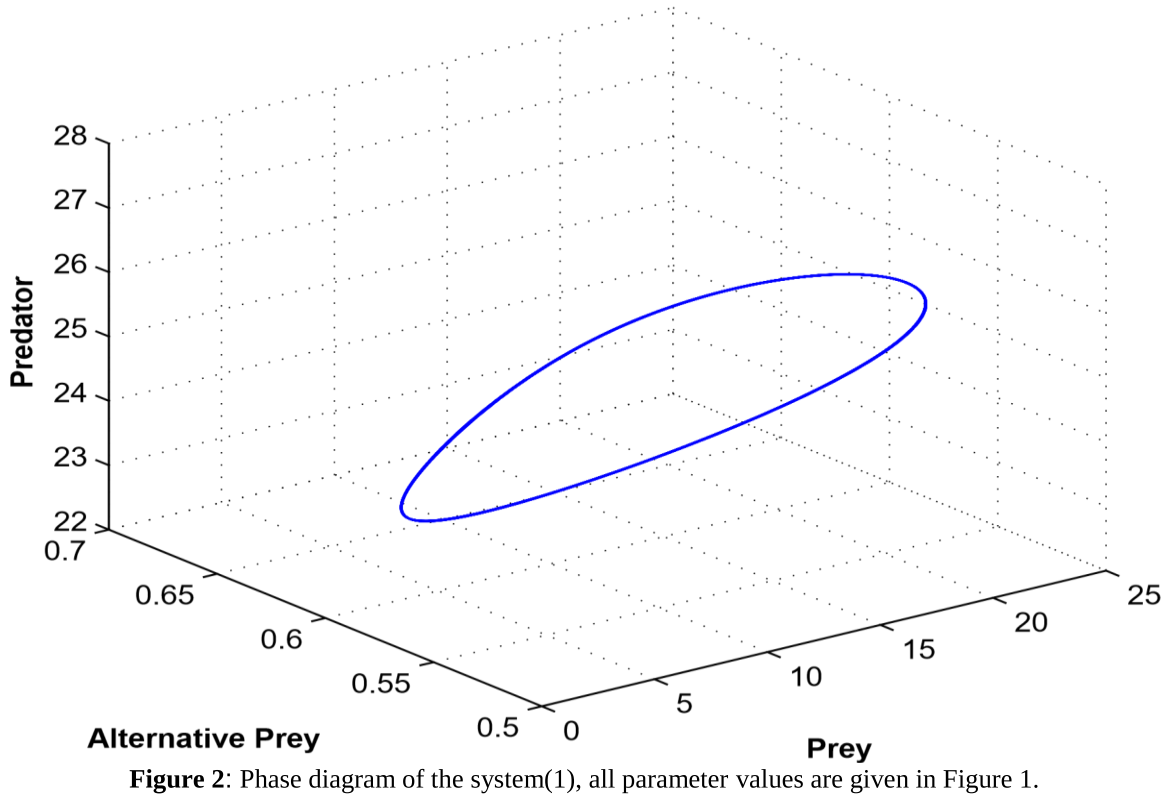

Figure 2: Phase diagram of the system(1), all parameter values are given in Figure 1.

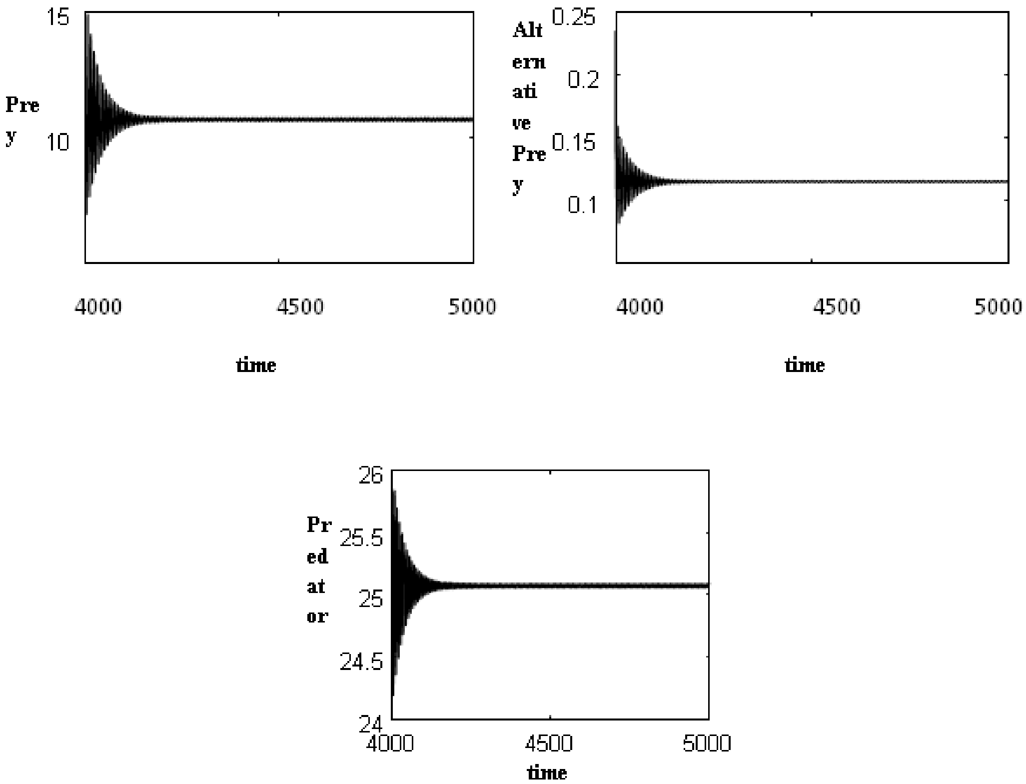

Figure 3: The figure shows the stable oscillation of the system(1) for $a_2 = 0.56$ and other parameter values are given in Figure 1.

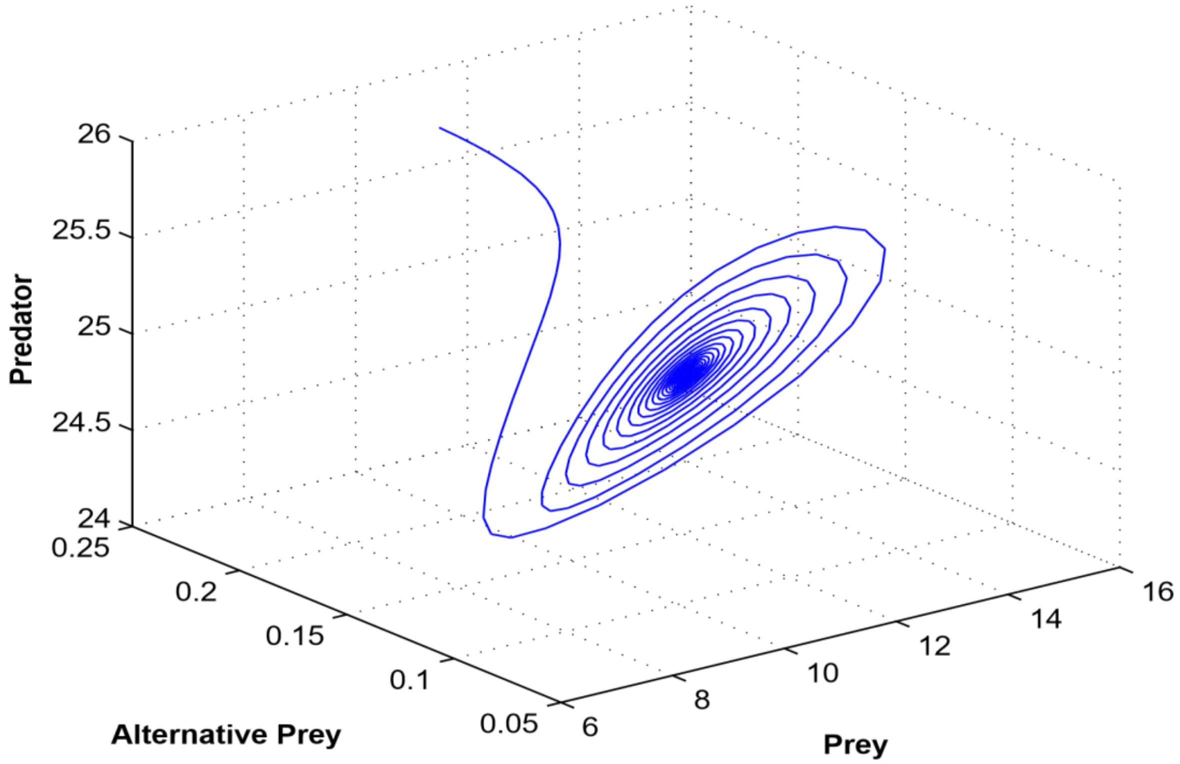

Figure 4: Phase diagram of the system(1) with respect to Figure 3.

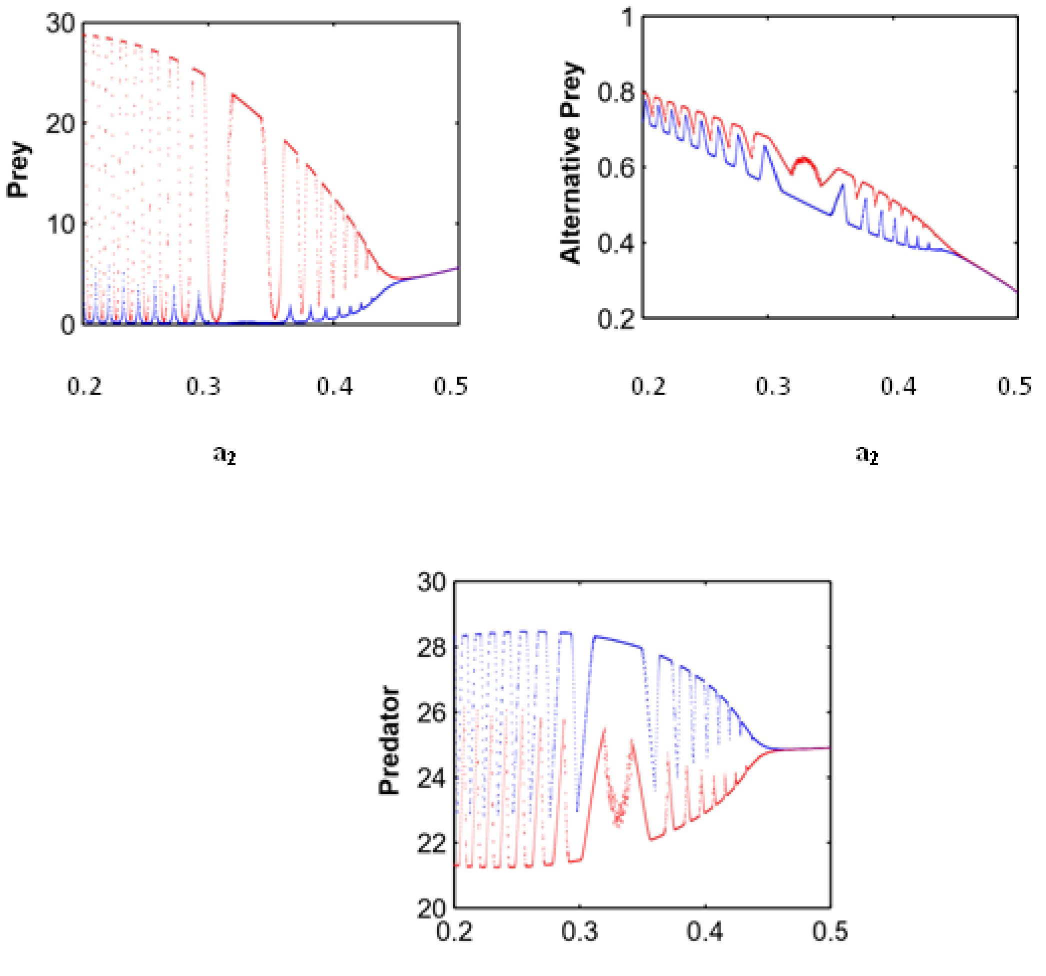

Figure 5: Figure shows the bifurcation diagram for $a_2 \in [0.2, 0.5]$ and other parameter values given in Figure-1

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Prodip Roy. “Role of Alternative Prey in A Predator-Prey Model With Prey Refugia - A Mathematical Study.” IOSR Journal of Engineering (IOSRJEN), vol. 09, no. 05, 2019, pp. 19-26.