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Fuzzy Partial Isometry Operator and Its Characteristics

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Abstract: In this paper, the definition of Fuzzy Partial Isometry operator on a FuzzyHilbert space(FH-space) is introduced. We have given many definitions, theorems and discuss in detail, some characteristics of Fuzzy Partial Isometry operator in FH-space.

Keywords: Fuzzy Unitary operator,Fuzzy Normal operator, Self-Adjoint Fuzzy operator, Fuzzy Partial Isometry operator (FPI - operator), FuzzyProjection operator.

I. INTRODUCTION

In 1984, first Katsaras[5] wasintroduced theconcept of Fuzzy norm on a linear space. Then after Felbin[7],Cheng and Mordeson[14], Samanta[17]etc.,have been given different dimensions in definition of fuzzy normed spaces. The definition of Fuzzy Inner Product space(FIP-space) first introduced by Biswas[12] and after that according the annals in [8],[9],[13],[6],[11]. Goudarzi and Vaezpour in [10],and Yongfusu[16] have beenmodified and included the definition of Fuzzy Inner Product space (FIP-space).Goudarzi and Vaezpour[9] has been introduced the definition of Fuzzy Hilbert space in 2009.Also,the implication of Fuzzy Hilbert space(FH-space) was given in [10]and [15].

Noori FAI- Mayahi and Abbas MAbbas[19] have given some properties of Spectral Theory of Linear Operator defined on fuzzy normed spaces which was considered as an expansion for the spectral theory of linear operator defined on normed spaces in 2016.SudadMRasheed [4]was introduced the concept and some properties of Adjoint Fuzzy Linear operators, self - adjoint Fuzzy Linear operators in 2017.Radharamani etal.[1] defined concept of fuzzy normal operator and their properties. Whereas the definition of fuzzy unitary operators and their properties has been given byRadharamani et al. [2] in 2018.

In this paper, we consider self-adjoint operator in FH-space and introduce the definitions of Fuzzy Partial Isometry operator, we establish some characteristics ofFuzzy Partial Isometry from Fuzzy operator in FH-space.

The arrangementof this paper is as follows:
Section 2 provides some preliminary definitions, resultsand theorems are used in this paper.
In section 3, we introduced the idea of FuzzyPartial Isometry operator, several theorems.
We discussedsuch Fuzzy operators.

II. PRELIMINARIES

Definition 2.1:[10]FIP-Space

A fuzzy inner product space (FIP-Space) is a triplet (V,F,*),where X is a real vector space,* is a continuous t-norm, F is a fuzzy set on $V^2\times R$ satisfying the following conditions for every x,y,z ∈ V and s,r,t ∈ R.

F1-1:F(x,x,0)=0 and F(x,x,t) > 0, for each t > 0.
F1-2:F(x,x,t)≠ H(t) for some t € R if ans only if x ≠ 0,
where $H(t) = \begin{cases} 1, if\ t > 0 \\ 0, if\ t \le 0 \end{cases}$
F1-3:F(x,y,t) = F(y,x,t)
F1-4:For any $\alpha \in$ R,F $(\alpha x, y, t) = \begin{cases} F(x, y, \frac{t}{\alpha})\alpha > 0 \\ H(t)\alpha = 0 \\ 1 - F(x, y, \frac{t}{-\alpha})\alpha < 0 \end{cases}$
F1-5:F(x, x,t)* F(y,y, s) ≤ F(x+y,x+y, t+s)
F1-6:$SUP_{s+r=t}$ [F(x,z,s)* F(y,z,r)] = F (x+y,z, t)
F1-7:F(x,y,.): R → [0,1] is continuous on R\ {0}
F1-8: $\lim_{t\to\infty} F(x, y, t) = 1$.

Definition 2.2: [10] Fuzzy Hilbert space (FH-space)

Let (V,F,*) be a FIP-space with IP $\langle x, y\rangle$ = sup {t ∈ R : F(x, y,t) <1},∀x,y ∈ V. If V is complete in the ‖.‖,then V is called Fuzzy Hilbert space(FH-space).

Definition 2.3: [4] Adjoint Fuzzy operator

Let (X,G,*) be a FH-space and let S ∈ FB(X) be T_F continuous liner functional.Then,∃ unique S*∈ FB(E) such that $\langle Su, v\rangle = \langle u, S * v\rangle$ ∀ u,v ∈ X.

Definition2.4: [4] Self-Adjoint Fuzzy operator

Let (X,G,*) be a FH-space with IP: $\langle u, v\rangle$ = sup {x ∈ R : G(u, v,x) <1},∀ u, v ∈ Xand let S ∈ FB(X).Then S is self-adjoint Fuzzy operator,if S =S*,where S* is adjoint Fuzzy operator of S.

Definition 2.5: [1] Fuzzy Normal operator

Let (X,F,*) be an FH-space with IP:$\langle u, v\rangle$ = sup {u ∈ R : F(u, v,x) <1},∀ u, v ∈ X and let S∈ $FB(H)$ then S is said to be an Fuzzy Normal operator if it commutes with its (fuzzy) adjoint i.e.SS*=S*S.

Definition2.6:[2] Fuzzy Unitary operator

Let U ∈ FB(H) is said to be a fuzzy unitary operator if U is a Fuzzy isometry operator from H onto H.

Definition 2.7: [2] Fuzzy isometry operator

Let (H,F,*) be an FH-space with IP:$\langle x, y\rangle$ = sup {u ∈ R : F(x, y,u) <1},∀ x,y ∈ H and let an operator U on a Fuzzy Hilbert space H.i.e.U ∈ FB(H) then U is said to be a Fuzzy isometry operator if $\|Ux\| = \|x\|$ for any x ∈ H. i.e.$\langle Ux, Uy\rangle = \langle x, y\rangle$

Definition2.8: [19]Fuzzy Projection operator

A Fuzzy Hilbert space H can be decomposed into H = M $\oplus M^{\perp}$ for any x ∈ H, x = y $\oplus z$, where y ∈ M and z ∈ $M^{\perp}$.Let Px = y.This transformation P defines a fuzzy linear operator from H onto M.This P is said to be fuzzyprojection of H onto M, denoted by P_M.

III. MAIN RESULTS

In this section, we introduce the definition of Fuzzy Partial Isometry operator in FH-space as well as some elementary properties of Fuzzy Partial Isometry operator in FH-space are presented.

Definition3.1: Fuzzy Partial isometry operator

Let $U \in FB(H)$is said to be a FuzzyPartial Isometry operator if ∃ closed subspace M, ∋ $\|Ux\| = \|x\|$for any $x \in M$ &$Ux = 0$,for any $x \in M^{\perp}$, where M is said to be the initial space of U and $N=R(U)$ is said to be the finial space of U.

Note:

1) The Fuzzy Projection onto the initial space and the final space are said to be the Initial Fuzzy Projection and Final Fuzzy projection of U.

2) U is Fuzzy Isometry iff U is Fuzzy Partial Isometry and $M = H$.

3) U is Fuzzy Unitary iff$U$ is Fuzzy Partial Isometry and $M = N = H$.

Theorem3.2:

Let $U \in FB(H)$ is aFuzzy Isometry operator iff$U^*U = I$.

Proof:

Assume that U is a Fuzzy Isometry,

$\langle U^*Ux, y\rangle$= *sup{u ∈ R : F(U*Ux, y,u) <1}*

= sup {u ∈ R : F(Ux, Uy,u) <1}

= sup {u ∈ R : F(x, y,u) <1}

$\langle U^*Ux, y\rangle = \langle x, y\rangle \ \forall\, x, y \in H$

$\Rightarrow U^*U = I$

Conversely, suppose that $U^*U = I$

$\|Ux\|^2 = \langle Ux, Ux\rangle$

= sup{u ∈ R : F(Ux, Ux,u) <1}

*= sup{u ∈ R : F(U*Ux, x,u) <1}*

= sup{u ∈ R : F(x,x,u) <1}

$= \langle x, x\rangle$

$\|Ux\|^2 = \|x\|^2 \Rightarrow \|Ux\| = \|x\|$

Hence U is a Fuzzy Isometry operator.

Theorem 3.3:

An operator $U \in FB(H)$ is Fuzzy Unitary operator iff$U^*U=UU^* = I$.

Proof:

If U is Fuzzy Unitary iff$U$ is a Fuzzy Isometry operator from H onto H.

$U^*U = I$ and for any $x \in H$,∃ $y \in H$,∋ $Uy = x$.

$U^*x = U^*Uy = y$

So that $\|U^*x\| = \|y\|$

$$= \|Uy\|$$
$$\|U^*x\| = \|x\|$$

Thus U^* is Fuzzy Isometry and

$UU^* = (U^*)^*U^* = I$

Conversely, assume that $U^*U = UU^* = I$

Then U is Fuzzy Isometry and for any $x \in H$, $x = UU^*$, $x \in R(U)$, where $R(U)$ isthe range of U.

So, U is Fuzzy Isometry operator from H onto H.

Theorem 3.4:

Let U be a Fuzzy Partial Isometry operator on a Fuzzy Hilbertspace with the initial space M and the final space N then the following (1),(2),(3) hold.

1) $UP_M = U$ *and* $U^*U = P_M$

2) N is a closed subspace of H.

3) U^* is a Partial Isometry with the initial space N and the final space M,that is, $U^*P_N = U^*$ and $UU^* = P_N$.

Proof:

Given that $U$is a Fuzzy Partial Isometry operator on FH-space.

To prove:(1) U = UP_M & $U^*U = P_M$ hold

For any $x \in H$

$x = P_M x \oplus z$ for $z \in M^{\perp}$

& $Ux = UP_M x \oplus Uz = UP_M x$

$U = UP_M \because U_Z = 0$

Hence $U = UP_M$.

As $\langle Ux, Uy\rangle = \langle x, y\rangle$ for $x,y \in M$ and $P_M x$, $P_M y \in M$ for any $x,y \in H$,

$\langle U^*Ux, y\rangle = sup\{u \in R : F(U^*Ux, y,u) < 1\}$

$= sup\{u \in R : F(Ux, Uy,u) < 1\}$

$= sup\{u \in R : F(UP_M x, UP_M y,u) < 1\}$

$= sup\{u \in R : F(U^*UP_M x, P_M y,u) < 1\}$

$= sup\{u \in R : F(P_M x, P_M y,u) < 1\}$

$= sup\{u \in R : F(P_M x, y,u) < 1\}$

$$\langle U^*Ux, y\rangle = \langle P_M x, y\rangle$$

$\Rightarrow U^*U = P_M$

(2) To prove N is a closed subspace of H.

As $N = R(U) = UR(P_M) = UM$, for any $x \in \overline{N}$,there exists a sequence $\{y_n\} \subset M$ such that $Uy_n \to x$ and $\|y_m - y_n\| = \|Uy_m - Uy_n\| \to 0$ as $m,n \to \infty$.

Thus, by the completeness of H,$\exists\, y \in H$, $\ni y_n \to y$ and $Uy_n \to Uy$ implies $x = Uy \in N$,

Hence $\overline{N} = N$.

(3) To Prove that U^* is Fuzzy Partial Isometry with the initial space N and the final space M.

For any $x \in N$,$\exists y \in M$, $\ni Uy = x$ and $\|x\| = \|y\|$, and

$U^*x = U^*Uy = P_M y = y$

So that $\|U^*x\| = \|x\|$. --------------(a)

for any $x \in N^{\perp}$, Since $Uy \in N$ for any $y \in H$,

$\Rightarrow \langle U^*x, y\rangle = \langle x, Uy\rangle = 0$

$\Rightarrow U^*x = 0$ --------------(b)

Therefore U^* is Fuzzy Partial Isometry with the initial space N and the final space M because

$R(U^*) = U^*N = U^*R(U) = U^*UH = P_M H = M$.

$U^* P_N = U^*$ and $UU^* = P_N$ follows from (1) by replacing U by U^* and M by N.

Theorem 3.5:

Let U be an operator on a FH –space H.Then the following statements are mutually equivalent.

(1) U is a Fuzzy Partial Isometry operator.

(1a) U^* is a Fuzzy Partial Isometry operator

(2) $UU^*U = U$

(2a) $U^*UU^* = U^*$

(3) U^*U is a Fuzzy Projection operator.

(3a) UU^* is a Fuzzy Projection operator.

Proof:

Let U be an operator on a FH-space.

(1)⇒ (2)

If U is a Fuzzy Partial Isometry operator then by theorem 3.3(1)

$UU^*U = U P_M = U$ $[UP_M = U$ and $U^*U = P_M]$

$\Rightarrow UU^*U = U$

Hence (1) ⇒ (2)

(2) ⇒ (3)

If $UU^*U = U$ then to prove that U^*U is a fuzzy projection operator.

$U^* UU^*U = U^*U$ by (2)

i.e. U^*U is idempotent and Self-adjoint,so that U^*U is a Fuzzy Projection operator.

Hence (2) ⇒ (3)

(3) ⇒ (1)

Suppose U^*U is a Fuzzy Projection operator,

Put$U^*U = P_M$.

For any $x \in H$.

$\|Ux\|^2 = \langle Ux, Ux \rangle$

$= \sup \{u \in R : F(Ux, Ux,u) < 1\}$

$= \sup \{u \in R : F(U^*Ux, x,u) < 1\}$

$= \sup \{u \in R : F(P_M x, x,u) < 1\}$

$= \|P_M x\|^2$

So that $\|Ux\| = \|x\|$ for any $x \in M$,

and $Ux = 0$ for any $x \in M^\perp$

Hence the proof of the equivalence relation among (**1**),(**2**)and (**3**) is complete.

Similarly, the proof of all equivalence relation among (**1a**),(**2a**),(**3a**) is easily shown and (**2**)↔(**2a**) is obtained by taking adjoint of both sides.

IV. CONCLUSION

As the concept of Fuzzy Partial Isometry operator in FH- space is comparatively new and typical form of the theorems play the great role in our paper of this discussion. Some concepts and properties have been investigated aboutFuzzy partial isometry operator in Fuzzy Hilbert space. The consequence of this paper will be favorable for researchers to establish Fuzzy Functional Analysis.

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